Introduction to Theory of Mesoscopic Systems

Boris Altshuler

Princeton University, Columbia University & NEC Laboratories America

Lecture 3
Beforehand: Weak Localization and Mesoscopic Fluctuations

Today: Random Matrices, Anderson Localization, and Quantum Chaos

Later: Interaction between electrons in mesoscopic systems
E.P. Wigner, Conference on Neutron Physics by Time of Flight, November 1956


**RANDOM MATRIX THEORY**

\[ \mathbf{N} \times \mathbf{N} \quad \text{ensemble of Hermitian matrices with random matrix element} \quad \mathbf{N} \rightarrow \infty \]

\[ E_\alpha \]

\[ \delta_1 \equiv \langle E_{\alpha+1} - E_\alpha \rangle \]

\[ \langle \ldots \ldots \rangle \]

\[ s \equiv \frac{E_{\alpha+1} - E_\alpha}{\delta_1} \]

\[ P(s) \]

- spectrum (set of eigenvalues)
- mean level spacing
- ensemble averaging
- spacing between nearest neighbors
- distribution function of nearest neighbors spacing between

**Spectral Rigidity**

**Level repulsion**

\[ P(s = 0) = 0 \]

\[ P(s \ll 1) \propto s^\beta \quad \beta = 1, 2, 4 \]
Noncrossing rule (theorem) \[ P(s = 0) = 0 \]

Suggested by Hund \((Hund F. 1927~Phys.~v.40,~p.742)\)


Usually textbooks present a simplified version of the justification due to Teller \((Teller~E.,~1937~J.~Phys.~Chem~41~109)\).

Arnold V. I., 1972 Funct. Anal. Appl. v. 6, p.94

In general, a multiple spectrum in typical families of quadratic forms is observed only for two or more parameters, while in one-parameter families of general form the spectrum is simple for all values of the parameter. Under a change of parameter in the typical one-parameter family the eigenvalues can approach closely, but when they are sufficiently close, it is as if they begin to repel one another. The eigenvalues again diverge, disappointing the person who hoped, by changing the parameter to achieve a multiple spectrum.
**RANDOM MATRICES**

\[ N \times N \] matrices with random matrix elements. \[ N \to \infty \]

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**Dyson Ensembles**

<table>
<thead>
<tr>
<th>Matrix elements</th>
<th>Ensemble</th>
<th>( \beta )</th>
<th>realization</th>
</tr>
</thead>
<tbody>
<tr>
<td>real</td>
<td>orthogonal</td>
<td>1</td>
<td>T-inv potential</td>
</tr>
<tr>
<td>complex</td>
<td>unitary</td>
<td>2</td>
<td>broken T-invariance (e.g., by magnetic field)</td>
</tr>
<tr>
<td>2 \times 2 matrices</td>
<td>simplectic</td>
<td>4</td>
<td>T-inv, but with spin-orbital coupling</td>
</tr>
</tbody>
</table>
Gaussian Orthogonal Ensemble (GOE)

Poisson

Orthogonal $\beta=1$

Unitary $\beta=2$

Simplectic $\beta=4$

Poisson – completely uncorrelated levels
Main goal is to classify the eigenstates in terms of the quantum numbers

For the nuclear excitations this program does not work

_Study spectral statistics of a particular quantum system - a given nucleus_
Main goal is to classify the eigenstates in terms of the quantum numbers.

For the nuclear excitations this program does not work.

**E.P. Wigner:** Study spectral statistics of a particular quantum system - a given nucleus.

<table>
<thead>
<tr>
<th>Random Matrices</th>
<th>Atomic Nuclei</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Ensemble</td>
<td>• <em>Particular quantum system</em></td>
</tr>
<tr>
<td>• Ensemble averaging</td>
<td>• <em>Spectral averaging</em> (over α)</td>
</tr>
</tbody>
</table>

Nevertheless, statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics.
Particular nucleus

$^{166}\text{Er}$

Spectra of several nuclei combined (after spacing) rescaling by the mean level

N. Bohr, Nature 137 (1936) 344.
Q: Why the random matrix theory (RMT) works so well for nuclear spectra

Original answer: These are systems with a large number of degrees of freedom, and therefore the “complexity” is high.

Later it became clear that there exist very “simple” systems with as many as 2 degrees of freedom (d=2), which demonstrate RMT-like spectral statistics.
Classical ($\hbar = 0$) Dynamical Systems with $d$ degrees of freedom

Integrable Systems

The variables can be separated and the problem reduces to $d$ one-dimensional problems

$d$ integrals of motion

Examples

1. A ball inside rectangular billiard; $d = 2$
   - Vertical motion can be separated from the horizontal one
   - Vertical and horizontal components of the momentum, are both integrals of motion

2. Circular billiard; $d = 2$
   - Radial motion can be separated from the angular one
   - Angular momentum and energy are the integrals of motion
### Classical Dynamical Systems with $d$ degrees of freedom

#### Integrable Systems

- The variables can be separated $\Rightarrow d$ one-dimensional problems $\Rightarrow d$ integrals of motion

- Rectangular and circular billiard, Kepler problem, . . . , 1d Hubbard model and other exactly solvable models, . . .

#### Chaotic Systems

- The variables cannot be separated $\Rightarrow$ there is only one integral of motion - energy

### Examples

- Sinai billiard
- Stadium
- Kepler problem in magnetic field

$B$
Classical Chaos
\[ \hbar = 0 \]

• Nonlinearities
• Exponential dependence on the original conditions (Lyapunov exponents)
• Ergodicity

Quantum description of any System with a finite number of the degrees of freedom is a linear problem - Shrödinger equation

Q: What does it mean *Quantum Chaos*?
Bohigas – Giannoni – Schmit conjecture

Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France
(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai’s billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal–invariant systems whose classical analogs are \( K \) systems show the same fluctuation properties as predicted by GOE.

\( \hbar \neq 0 \)

No quantum numbers except energy

Chaotic classical analog

Wigner- Dyson spectral statistics
Q: What does it mean Quantum Chaos?

Two possible definitions

Chaotic classical analog

Wigner - Dyson-like spectrum
Quantum

Integrable

Classical

Poisson

Wigner-Dyson

Chaotic

? 

?
Important example: quantum particle subject to a random potential - disordered conductor

- Scattering centers, e.g., impurities

- As well as in the case of Random Matrices (RM) there is a luxury of ensemble averaging.

- The problem is much richer than RM theory

- There is still a lot of universality.

Anderson localization (1958)

At strong enough disorder all eigenstates are localized in space
Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential - disordered conductor

Scattering centers, e.g., impurities

Models of disorder:

Randomly located impurities
White noise potential
Lattice models
Anderson model
Lifshits model
Anderson Model

- Lattice - tight binding model
- Onsite energies $\varepsilon_i$ - random
- Hopping matrix elements $I_{ij}$

$-W < \varepsilon_i < W$ uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$I < I_c$

Insulator

All eigenstates are localized

Localization length $\xi$

$I > I_c$

Metal

There appear states extended all over the whole system
**Anderson Transition**

**Strong disorder**

\[ I < I_c \]

*Insulator*

All eigenstates are **localized**

Localization length \( \xi \)

The eigenstates, which are localized at different places will not repel each other

*Poisson spectral statistics*

**Weak disorder**

\[ I > I_c \]

*Metal*

There appear states **extended** all over the whole system

Any two extended eigenstates repel each other

*Wigner – Dyson spectral statistics*
Zharekeshev & Kramer.

Exact diagonalization of the Anderson model
Critical electron eigenstate at the Anderson transition

100 × 100 × 100
Anderson model cube

Zharekeshev, Computer Phys. Commun. 1999
Energy scales (Thouless, 1972)

1. Mean level spacing

\[ \delta_1 = \frac{1}{\nu} \times L^d \]

\( L \) is the system size;

\( d \) is the number of dimensions

2. Thouless energy

\[ E_T = \frac{hD}{L^2} \]

\( D \) is the diffusion constant

\( E_T \) has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)

\[ g = \frac{E_T}{\delta_1} \]

\( g \) is dimensionless Thouless conductance

\[ g = \frac{Gh}{e^2} \]
Thouless Conductance and One-particle Spectral Statistics

Localized states
Insulator
Poisson spectral statistics

Extended states
Metal
Wigner-Dyson spectral statistics

Transition at $g \sim 1$.
Is it sharp?
Conductance $g$

$100 \times 100 \times 100$

Anderson model cube
Anderson transition in terms of pure level statistics

\[ P(s) \]

- **metal, W=5**
- **critical, 16.5**
- **insulator, 100**

- Wigner
- Poisson

Scaling of level spacing variance

Linear size of 3D cube

GOE

disorder W
\[ \frac{d \log g}{d \log L} = \beta(g) \]

\( \beta - \text{function} \)

Metal – insulator transition in 3D
All states are localized for \( d=1,2 \)
Thouless Conductance and One-particle Spectral Statistics

Localized states
Insulator
Poisson spectral statistics

Extended states
Metal
Wigner-Dyson spectral statistics

\( N \times N \) Random Matrices

Quantum Dots with Thouless conductance \( g \)

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

\( N \to \infty \)

\( g \to \infty \)
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar

Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

Integrable

All chaotic systems resemble each other.

Chaotic

All integrable systems are integrable in their own way
Disordered Systems:

- Anderson metal;
- Wigner-Dyson spectral statistics
- Anderson insulator;
- Poisson spectral statistics

Q: Is it a generic scenario for the Wigner-Dyson to Poisson crossover?

Speculations

Consider an integrable system. Each state is characterized by a set of quantum numbers.

It can be viewed as a point in the space of quantum numbers. The whole set of the states forms a lattice in this space.

A perturbation that violates the integrability provides matrix elements of the hopping between different sites (Anderson model !?)
Consider an **integrable** system. Each state is characterized by a **set of quantum numbers**.

It can be viewed as a point in the **space of quantum numbers**. The whole set of the states forms a **lattice** in this space.

A **perturbation** that violates the integrability provides matrix elements of the **hopping** between different sites (**Anderson model!**?)

**Weak enough hopping** - Localization - **Poisson**

**Strong hopping** - transition to **Wigner-Dyson**
The very definition of the localization is not invariant – one should specify in which space the eigenstates are localized.

Level statistics is invariant:

- Poissonian statistics exists in a basis where the eigenfunctions are localized.
- Wigner-Dyson statistics holds in a basis the eigenfunctions are extended.
Example 1

**Doped semiconductor**

- Low concentration of donors → Electrons are localized on donors ⇒ Poisson
- Higher donor concentration → Electronic states are extended ⇒ Wigner-Dyson
**Example 1**

Doped semiconductor

- **Low concentration of donors** → Electrons are localized on donors ⇒ Poisson
- **Higher donor concentration** → Electronic states are extended ⇒ Wigner-Dyson

**Example 2**

Rectangular billiard

- **Two integrals of motion**
  
  \[ p_x = \frac{\pi n}{L_x}; \quad p_y = \frac{\pi m}{L_x} \]

- Lattice in the momentum space
- Line (surface) of constant energy
- **Ideal billiard**
  - Localization in the momentum space ⇒ Poisson
- **Deformation or smooth random potential**
  - Delocalization in the momentum space ⇒ Wigner-Dyson

Ideal billiard:

- Localization in the momentum space ⇒ Poisson
- Delocalization in the momentum space ⇒ Wigner-Dyson
Localization and diffusion in the angular momentum space

Diffusion and Localization in Chaotic Billiards

Fausto Borgonovi, Giulio Casati, Baowen Li

1 Dipartimento di Matematica, Università Cattolica, via Trieste 17, 25121 Brescia, Italy
2 Università di Milano, sede di Como, Via Luria 3, Como, Italy
3 Istituto Nazionale di Fisica della Materia, Unità di Milano, via Celoria 16, 20133, Milano, Italy
4 Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, Pavia, Italy
5 Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Milano, Italy
6 Department of Physics and Centre for Nonlinear and Complex Systems, Hong Kong Baptist University, Hong Kong
7 Center for Applied Mathematics and Theoretical Physics, University of Maribor, Kromer 2, 2000 Maribor, Slovenia

(Received 29 July 1996)

\[ \varepsilon \equiv \frac{a}{R} \]

\[ \varepsilon > 0 \text{ Chaotic stadium} \]

\[ \varepsilon \rightarrow 0 \text{ Integrable circular billiard} \]

Angular momentum is the integral of motion

\[ \hbar = 0; \quad \varepsilon \ll 1 \]

Angular momentum is not conserved
Localization and diffusion in the angular momentum space

**Diffusion and Localization in Chaotic Billiards**

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\[ \varepsilon \equiv \frac{a}{R} \]

**Chaotic stadium**

**Integrable circular billiard**

\[ \varepsilon \to 0 \]

Angular momentum is the integral of motion

\[ \frac{dL}{dt} = 0 \]

**Diffusion in the angular momentum space**

\[ D \propto \varepsilon^{5/2} \]

**Poisson**

\[ \varepsilon = 0.01 \]
\[ g = 0.012 \]

**Wigner-Dyson**

\[ \varepsilon = 0.1 \]
\[ g = 4 \]
1D Hubbard Model on a periodic chain

\[ H = t \sum_{i,\sigma} \left( c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma} \right) + U \sum_{i,\sigma} n_{i,\sigma} n_{i,-\sigma} + V \sum_{i,\sigma,\sigma'} n_{i,\sigma} n_{i+1,\sigma'} \]

- \( V = 0 \): Hubbard model, integrable
- \( V \neq 0 \): extended Hubbard model, nonintegrable

\( V = 0 \) corresponds to the integrable Hubbard model.
\( V \neq 0 \) corresponds to the nonintegrable extended Hubbard model.

Onsite interaction
n. neighbors interaction
1D Hubbard Model on a periodic chain

\[ H = t \sum_{i, \sigma} \left( c_{i,\sigma}^+ c_{i+1,\sigma} + c_{i+1,\sigma}^+ c_{i,\sigma} \right) + U \sum_{i, \sigma} n_{i,\sigma} n_{i,-\sigma} + V \sum_{i, \sigma, \sigma'} n_{i,\sigma} n_{i+1,\sigma'} \]

\( V = 0 \) Hubbard model integrable

\( V \neq 0 \) extended Hubbard model nonintegrable

12 sites
3 particles
Zero total spin
Total momentum \( \pi/6 \)
1D $t$-$J$ model on a periodic chain

Exchange

Hopping
1D $t$-$J$ model on a periodic chain

1d $t$-$J$ model
1D $t$-$J$ model on a periodic chain

$J=t$

$J=2t$

$J=5t$

$N=16$; one hole
Why the random matrix theory (RMT) works so well for nuclear spectra
Chaos in Nuclei – Delocalization?

Delocalization in Fock space

Fermi Sea