Diffusion Equation

\[ \frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0 \]

Lessons from the Einstein’s work:

- **Universality**: the equation is valid as long as the process is markovian

- Can be applied to the **probability** and thus describes both fluctuations and dissipation

- There is a universal relation between the diffusion constant and the viscosity

- Studies of the diffusion processes brings information about micro scales.
Lesson 1:

Beyond Markov chains:

Anderson Localization

and

Magnetoresistance
Energy scales (Thouless, 1972)

1. Mean level spacing

\[ \delta_1 = \frac{1}{\nu} \times L^d \]

- \( L \) is the system size;
- \( d \) is the number of dimensions

2. Thouless energy

\[ E_T = \frac{hD}{L^2} \]

- \( D \) is the diffusion constant

\( E_T \) has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)

\[ g = \frac{E_T}{\delta_1} \]

- dimensionless Thouless conductance

\[ g = \frac{Gh}{e^2} \]
$\beta$ - function

\[ \frac{d \log g}{d \log L} = \beta(g) \]

Metal – insulator transition in $3D$

All states are localized for $d=1,2$
Questions:

Why

• the scaling theory is correct?
• the corrections of the diffusion constant and conductance are negative?

Why diffusion description fails at large scales?
Diffusion description fails at large scales

Why?

Einstein: there is no diffusion at too short scales - there is memory, i.e., the process is not markovian.

Why there is memory at large distances in quantum case?

Quantum corrections at large Thouless conductance - weak localization
Universal description
\[ \beta(g) = d - 2 + \frac{c_d}{g} \]

\[ c_d = ? \quad \pm ? \]

\[ g(L) = \sigma_{cl} L^{d-2} - \frac{c_d}{d-2} \quad d \neq 2 \]

\[ c_2 \log \left( \frac{L}{l} \right) \quad d = 2 \]
Quantum corrections

\[ \beta(g) = d - 2 + \frac{c_d}{g} \]

\[ c_d = ? \pm ? \]

\[ g(L) = \sigma_c L^{d-2} - \frac{c_d}{d-2} \quad d \neq 2 \]

\[ c_2 \log \left( \frac{L}{l} \right) \quad d = 2 \]

Suggested homework:

1. Derive the equation for \( g(L) \) from this limit of the \( \beta \)-function.

2. Suppose you know \( \beta(g) \) for some number of dimensions \( d \). Let \( g \) at some size of the system \( L_0 \) be close to the critical value: \( g(L_0) = g_c + \delta g; \quad |\delta g| \ll 1 \). Estimate the localization length \( \xi \) (for \( \delta g < 0 \)) and the conductivity \( \sigma \) in the limit \( L \to \infty \) (for \( \delta g > 0 \)).
**Weak Localization**

Phase accumulated when traveling along the loop:

\[ \varphi = \oint \vec{p} \cdot d\vec{r} \]

The particle can go around the loop in two directions.

Memory!

Constructive interference probability to return to the origin gets enhanced diffusion constant gets reduced. **Tendency towards localization**

\[ \varphi_1 = \varphi_2 \]

\( \beta \)-function is negative for \( d=2 \)

The particle can go around the loop in two directions.
Diffusion

Random walk

Density fluctuations $\rho(r,t)$ at a given point in space $r$ and time $t$.

$$\frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0$$

Diffusion Equation

$D$ - Diffusion constant

Mean squared distance from the original point at time $t$

$$\langle r(t)^2 \rangle = Dt$$

Probability to come back (to the element of the volume $dV$ centered at the original point)

$$P(r(t) = 0) dV = \frac{dV}{(Dt)^{d/2}}$$
What is the probability $P(t)$ that such a loop is formed within a time $t$?

**Probability to come back (to the element of the volume $dV$ around the original point)**

$$P(r(t) = 0) dV = \frac{dV}{(Dt)^{d/2}}$$

**Q:** $dV = ?$

**A:** $dV = \lambda^{d-1} v_F dt$
What is the probability $P(t)$ that such a loop is formed within a time $t$?

**Q:** $dV = ?$

**A:** $dV = \hat{\lambda}^{d-1} v_F dt$

$$P(t) = -\hat{\lambda}^{d-1} \int_{\tau}^{t} \frac{v_F dt'}{(Dt')^{d/2}}$$

$$\frac{\delta g}{g} \approx P(t_{\text{max}})$$
\[ P(t) = -\dot{\lambda} d^{-1} \int_{\tau}^{t} \frac{v_F dt'}{(Dt')^{d/2}} \]

\[ \frac{\delta g}{g} \approx P(t_{\text{max}}) \]

**Q:** \( t_{\text{max}} = ? \)

**A:** \( t_{\text{max}} \sim \min \left\{ \frac{L^2}{D}, \frac{1}{\omega}, \tau_{\phi}, ... \right\} \)
\[ P(t) = -\hat{\lambda}^{d-1} \left( \int_{\tau}^{t} \frac{v_F dt'}{(D t')^{d/2}} \right) \]

\[ \frac{\delta g}{g} \approx P(t_{\text{max}}) \]

\[ t_{\text{max}} \sim \frac{L^2}{D} = \frac{h}{E_T} \]

\[ d = 2 \]

\[ \frac{\delta g}{g} \approx -\frac{\hat{\lambda} v_F}{D} \log \frac{L^2}{D \tau} \]
\[ P(t) = \tilde{\Lambda}^{d-1} \int_{\tau}^{t} \frac{v_F dt'}{(D t')^{d/2}} \]

\[ \frac{\delta g}{g} \approx P(t_{\text{max}}) \]

\[ \frac{\delta g}{g} \approx -\frac{\tilde{\Lambda}v_F}{D} \log \frac{L^2}{D \tau} = -\frac{2\tilde{\Lambda}v_F}{D} \log \frac{L}{l} \]

\[ \tilde{\Lambda}v_F = \frac{1}{\pi v} \]

\[ g = v D \hbar \]

\[ \delta g = -\frac{2}{\pi} \log \frac{L}{l} \]

\[ \beta(g) = -\frac{2}{\pi g} \]

Universal !!!
**Magnetoresistance**

No magnetic field

\[ \varphi_1 = \varphi_2 \]

With magnetic field \( \mathbf{H} \)

\[ \varphi_1 - \varphi_2 = 2 \times 2\pi \frac{\Phi}{\Phi_0} \]
Length Scales

Magnetic length

\[ L_H = \left( \frac{hc}{eH} \right)^{1/2} \]

Dephasing length

\[ L_\varphi = \left( D \tau_\varphi \right)^{1/2} \]

\[ \delta g(H) = f_d \left( \frac{L_H}{L_\varphi} \right) \]

Universal functions

Magnetoresistance measurements allow to study inelastic collisions of electrons with phonons and other electrons.
Weak Localization

Negative Magnetoresistance

Aharonov-Bohm effect

Theory
B.A., Aronov & Spivak (1981)

Experiment
Sharvin & Sharvin (1981)

Chentsov (1949)

FIG. 8. Longitudinal magnetoresistance $\Delta R(H)$ at $T=1.1$ K for a cylindrical lithium film evaporated onto a 1-cm-long quartz filament. $R_{d1}=2$ k$\Omega$, $R_{X1}/R_{X2}=2.8$. Solid line: averaged from four experimental curves. Dashed line: calculated for $L_0=2.2$ $\mu$m, $\tau_q/\tau_0=0$, filament diameter $d=1.31$ $\mu$m, film thickness 127 nm. Filament diameter measured with scanning electron microscope yields $d=1.30\pm0.03$ $\mu$m (Altshuler et al., 1982; Sharvin, 1984).
Q: What does it mean $d=2$?

A: Transverse dimension is much less than $\sqrt{Dt_{\text{max}}}$.
Lesson 2:

Brownian Particle as a mesoscopic system
Mesoscopic fluctuations

Magnetoresistance of small, quasi-one-dimensional, normal-metal rings and lines

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Yorktown Heights, New York 10598
(Received 6 July 1984)

The magnetoresistance of sub-0.4-μm-diam Au and Au$_{60}$Pd$_{40}$ rings was measured in a perpendicular magnetic field at temperatures as low as 5 mK in search of simple, periodic resistance oscillations that would be evidence of flux quantization in normal metal rings. The very complex structure developed in the magnetoresistance data did not reveal convincing evidence for flux quantization, but the oscillations that observed in the rings was also found in the lines. This structure appears to be associated with the onset of the superconducting transition.
Properties of systems with identical set of macroscopic parameters but different realizations of disorder are different!

\[ g_1 \neq g_2 \]
Properties of systems with identical set of macroscopic parameters but different realizations of disorder are different!

\[ g_1 \neq g_2 \]

Magnetoresistance \( g(H) \) is sample-dependent.

\[ \langle g \rangle \approx 1 \]

- ensemble averaging

\[ \langle g \rangle \ll 1 \]
Before Einstein:

Correct question would be: describe $\mathbf{r}(t)$

OK, maybe you can restrict yourself by $\langle \mathbf{r}(t) \rangle$
Before Einstein:

Correct question would be: describe $\vec{r}(t)$

OK, maybe you can restrict yourself by $\big< \vec{r}(t) \big>$

Einstein: What is $\big< \left[ \vec{r}(0) - \vec{r}(t) \right]^2 \big>$? 

$\big< \left[ \vec{r}(0) - \vec{r}(t) \right]^n \big> = ?$
Before Einstein:
Correct question would be: describe $\vec{r}(t)$

OK, maybe you can restrict yourself by $\langle \vec{r}(t) \rangle$

Einstein: What is $\langle [\vec{r}(0) - \vec{r}(t)]^2 \rangle$?

$\langle [\vec{r}(0) - \vec{r}(t)]^n \rangle = ?$

Mesoscopic physics: Not only $\langle g(H) \rangle$

But also $\langle [g(H) - g(H + h)]^2 \rangle$
<table>
<thead>
<tr>
<th></th>
<th><strong>Brownian motion</strong></th>
<th><strong>Conductance fluctuations</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>ensemble</td>
<td>Set of brownian particles</td>
<td>Set of small conductors</td>
</tr>
<tr>
<td>observables</td>
<td>Position of each particle $\vec{r}$</td>
<td>Conductance of each sample $g$</td>
</tr>
<tr>
<td>evolves as function of</td>
<td>Time $t$</td>
<td>Magnetic field $H$ or any other external tunable parameter</td>
</tr>
<tr>
<td>Interested in</td>
<td>Statistics of $\vec{r}(t)$</td>
<td>Statistics of $g(H)$</td>
</tr>
<tr>
<td>Example</td>
<td>$\left\langle \left[\vec{r}(t_1) - \vec{r}(t_2)\right]^2 \right\rangle$</td>
<td>$\left\langle \left[g(H_1) - g(H_2)\right]^2 \right\rangle$</td>
</tr>
</tbody>
</table>
Magnetoresistance

\[ g_1 - g_2 \approx 1 \]

\[ G_1 - G_2 \approx \frac{e^2}{\hbar} \]

Statistics of the functions of \( g(H) \) are universal

B.A. (1985); Lee & Stone (1985)
Statistics of random function(s) $g(H)$ are universal !!!

In particular,

$$\langle (\delta g)^2 \rangle \sim 1$$

$$g \propto L^{d-2} \rightarrow \frac{\langle (\delta g)^2 \rangle}{g^2} \propto L^{4-2d} \gg L^{-d}$$

Fluctuations are large and nonlocal
\[ W_1, W_2 \quad A_1, A_2 \]

probabilities\hspace{2cm} probability amplitudes

\[ W_{1,2} = |A_{1,2}|^2 \]

\[ A_{1,2} = |A_{1,2}| e^{i\phi_{1,2}} \]

Total probability

\[ W = |A_1 + A_2|^2 = W_1 + W_2 + 2 \text{Re}(A_1 A_2^*) \]

interference term:

\[ 2 \text{Re}(A_1 A_2^*) = 2 \sqrt{W_1 W_2} \cos(\phi_1 - \phi_2) \]
\[ W = |A_1 + A_2|^2 = W_1 + W_2 + 2 \text{Re}(A_1 A_2^*) \]

\[ 2 \text{Re}(A_1 A_2^*) = 2 \sqrt{W_1 W_2} \cos(\varphi_1 - \varphi_2) \]

1. \( A_{1,2} = \sqrt{W_{1,2}} \exp(i \varphi_{1,2}) \)

2. Phases \( \varphi_{1,2} \) are random

3. \( |\varphi_1 - \varphi_2| >> 2\pi \)

\[ \langle W \rangle = \langle W_1 \rangle + \langle W_2 \rangle \]

The interference term disappears after averaging

\[ \langle \cos(\varphi_1 - \varphi_2) \rangle = 0 \]
Classical result for average probability:

\[ \langle W \rangle = W_1 + W_2 \]
Consider now square of the probability

$$\langle W^2 \rangle = (W_1 + W_2)^2 + 2W_1 W_2$$

**Reason:**

$$\langle \cos(\phi_1 - \phi_2) \rangle = 0$$
$$\langle \cos^2(\phi_1 - \phi_2) \rangle = 1/2$$

$$\langle W^2 \rangle \neq \langle W \rangle^2$$
CONCLUSIONS:

1. There are fluctuations!
2. Effect is nonlocal.
Now let us try to understand the effect of magnetic field. Consider the correlation function

\[
\langle W(H)W(H+h) \rangle = \langle W(H) \rangle \langle W(H+h) \rangle + 2W_1W_2 \langle \cos(\delta\varphi(H)) \cos(\delta\varphi(H+h)) \rangle
\]

\[\delta\varphi \equiv \varphi_1 - \varphi_2\]

\[
\langle \cos(\delta\varphi(H)) \cos(\delta\varphi(H+h)) \rangle \Rightarrow \frac{1}{2} \quad \text{for } h \to 0 \quad (\Phi(h) \ll \Phi_0)
\]

\[
0 \quad \text{for } \Phi(h) \gg \Phi_0
\]

\[\Phi(h) = h \cdot (\text{area of the loop})\]
Magnetoresistance

\[ g(H) \]

\[ \langle g \rangle \]

Flux through the whole system

\[ \Phi_0 \]
Quantum Chaos

Marcus et al
Marcus et al., 1998

1. Disorder \((\times - \text{impurities})\)

2. Complex geometry

How to deal with disorder?

- Solve the Shrodinger equation exactly
- Make statistical analysis

What if there is no disorder?
Part 2:
Random Matrix Theory
And
Quantum Chaos
**RANDOM MATRIX THEORY**

\[ N \times N \] \textit{ensemble of Hermitian matrices with random matrix element} \[ \implies N \to \infty \]

- spectrum (set of eigenvalues)
- mean level spacing
- \textit{ensemble} averaging
- spacing between nearest neighbors
- distribution function of nearest neighbors spacing between

**Spectral Rigidity**

**Level repulsion**

\[ P(s = 0) = 0 \]

\[ P(s \ll 1) \propto s^\beta \quad \beta = 1, 2, 4 \]
Wigner-Dyson; GOE

Poisson

Gaussian Orthogonal Ensemble

Unitary $\beta=2$

Simplectic $\beta=4$

Poisson – completely uncorrelated levels
Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(H_{22} - H_{11})^2 + |H_{12}|^2}$$

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If $H_{12}$ is real (orthogonal ensemble), then for $s$ to be small two statistically independent variables ($H_{22} - H_{11}$ and $H_{12}$) should be small and thus $P(s) \propto s$, $\beta = 1$.

3. Complex $H_{12}$ (unitary ensemble) $\Rightarrow$ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent $\Rightarrow$ three independent random variables should be small $\Rightarrow P(s) \propto s^2$, $\beta = 2$.
Random Matrices

$N \times N$ matrices with random matrix elements. $N \to \infty$

Dyson Ensembles

<table>
<thead>
<tr>
<th>Matrix elements</th>
<th>Ensemble</th>
<th>$\beta$</th>
<th>realization</th>
</tr>
</thead>
<tbody>
<tr>
<td>real</td>
<td>orthogonal</td>
<td>1</td>
<td>T-inv potential</td>
</tr>
<tr>
<td>complex</td>
<td>unitary</td>
<td>2</td>
<td>broken T-invariance (e.g., by magnetic field)</td>
</tr>
<tr>
<td>$2 \times 2$ matrices</td>
<td>simplectic</td>
<td>4</td>
<td>T-inv, but with spin-orbital coupling</td>
</tr>
</tbody>
</table>
Finite size quantum physical systems

Atoms
Nuclei
Molecules

\{ \text{Quantum Dots} \}
Main goal is to classify the eigenstates in terms of the quantum numbers.

For the nuclear excitations this program does not work.

Study spectral statistics of a particular quantum system – a given nucleus.

<table>
<thead>
<tr>
<th>Random Matrices</th>
<th>Atomic Nuclei</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Ensemble</td>
<td>• Particular quantum system</td>
</tr>
<tr>
<td>• Ensemble averaging</td>
<td>• Spectral averaging (over $\alpha$)</td>
</tr>
</tbody>
</table>

Nevertheless, the statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics.
Particular nucleus

$^{166}_{\text{Er}}$

Spectra of several nuclei combined (after rescaling by the mean level spacing)
E.P. Wigner, Conference on Neutron Physics by Time of Flight, November 1956