Introduction to Theory of Mesoscopic Systems

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Introduction
Six papers:

1. The light-quantum and the photoelectric effect. 
   Completed March 17.

   Completed April 30. Published in 1906 Ph.D. thesis.

3. Brownian Motion. 
   Received by Annalen der Physik May 11.

4,5. The two papers on special relativity. 
   Received June 30 and September 27

   Received December 19.
Einstein’s Miraculous Year - 1905

Six papers:


3. Brownian Motion. Received by Annalen der Physik May 11.

4, 5. The two papers on special relativity. Received June 30 and September 27

6. Second paper on Brownian motion. Received December 19.
Diffusion and Brownian Motion:


3. Brownian Motion. Received by Annalen der Physik May 11.

6. Second paper on Brownian motion. Received December 19.

Q: Are these papers indeed important enough to stay in the same line with the relativity and photons. Why?
Einstein’s Miraculous Year - 1905

Six papers:


3. Brownian Motion. Received by Annalen der Physik May 11.

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6. Second paper on Brownian motion. Received December 19.

By far the largest number of citations

Nobel Prize
Brownian Motion - history

Robert Brown (1773-1858)

The instrument with which Robert Brown studied Brownian Motion and which he used in his work on identifying the nucleus of the living cell. This instrument is preserved at the Linnean Society in London.
Brownian Motion - History

Robert Brown, *Phil. Mag.* 4,161(1828); 6,161(1829)

Random motion of particles suspended in water (“dust or soot deposited on all bodies in such quantities, especially in London”)

Action of water molecules pushing against the suspended object

Giovanni Cantoni (Pavia). *N. Cimento*, 27,156(1867).
... measurements on the Brownian movement showed that Einstein's theory was in perfect agreement with reality. Through these measurements a new determination of Avogadro's number was obtained.

Jean Baptiste Perrin
France
b. 1870
d. 1942

The Nobel Prize in Physics 1926
From the Presentation Speech by Professor C.W. Oseen, member of the Nobel Committee for Physics of The Royal Swedish Academy of Sciences on December 10, 1926
Action of water molecules pushing against the suspend object

Problems:
1. Each molecule is too light to change the momentum of the suspended particle.
2. Does Brownian motion violate the second law of thermodynamics?

“We see under our eyes now motion transformed into heat by friction, now heat changes inversely into motion. This is contrary to Carnot's principle.”

Problems:

1. Each molecules is too light to change the momentum of the suspended particle.
2. Does Brownian motion violate the second law of thermodynamics?
3. Do molecules exist as real objects and are the laws of mechanics applicable to them?
Kinetic theory

\[ S = k \log W + \text{const} \]

entropy

probability

Ludwig Boltzmann
1844 - 1906
Kinetic theory

\[ S = k \log W + \text{const} \]

entropy \quad probability

\( k \) is Boltzmann constant

\[
\rho(\nu, T) = \frac{8\pi \hbar \nu^3}{c^3 \left[ \exp\left(\frac{\hbar \nu}{kT}\right) - 1 \right]}
\]

Ludwig Boltzmann
1844 - 1906

Max Planck
1858 - 1947
From Micro to Macro

\[ S = k \log W + \text{const} \]

From Macro to Micro
“It is of great importance since it permits exact computation of Avogadro number … . The great significance as a matter of principle is, however … that one sees directly under the microscope part of the heat energy in the form of mechanical energy.”

Einstein, 1915
Brownian Motion - history

Einstein was not the first to:

1. Attribute the Brownian motion to the action of water molecules pushing against the suspended object
2. Write down the diffusion equation
3. Saved Second law of Thermodynamics

L. Szilard, Z. Phys, 53, 840(1929)
Brownian Motion - history

Einstein was not the first to:

1. Attribute the Brownian motion to the action of water molecules pushing against the suspended object
2. Write down the diffusion equation

Einstein was the first to:

1. Apply the diffusion equation to the probability
2. Derive the diffusion equation from the assumption that the process is markovian (before Markov) and take into account nonmarkovian effects
3. Derived the relation between diffusion const and viscosity (conductivity), i.e., connected fluctuations with dissipation

By studying large molecules in solutions sugar in water or suspended particles Einstein made molecules visible
Diffusion Equation

\[ \frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0 \]

Diffusion constant

Einstein-Sutherland Relation for electric conductivity \( \sigma \)

\[ \sigma = e^2 D \nu \quad \nu \equiv \frac{dn}{d\mu} \]

If electrons would be degenerate and form a classical ideal gas

\[ \nu = \frac{1}{T n_{\text{tot}}} \]

William Sutherland (1859-1911)
Einstein-Sutherland Relation for electric conductivity $\sigma$

\[ n = n(\mu) \]

\[ \frac{dn}{dx} = \frac{dn}{d\mu} \frac{d\mu}{dx} = eE \frac{dn}{d\mu} \]

No current

Density of electrons

Chemical potential

Electric field

\[ eD \frac{dn}{dx} = \sigma E \]
Einstein-Sutherland Relation for electric conductivity $\sigma$

\[ n = n(\mu) \]
\[ \frac{dn}{dx} = \frac{dn}{d\mu} \frac{d\mu}{dx} = eE \frac{dn}{d\mu} \]

Electric field

Chemical potential

Density of states

Conductivity

Density of electrons

No current

Metal
Lessons from the Einstein’s work:

- **Universality**: the equation is valid as long as the process is markovian

- Can be applied to the probability and thus describes both fluctuations and dissipation

- There is a universal relation between the diffusion constant and the viscosity

- Studies of the diffusion processes brings information about micro scales.
What is a Mesoscopic System?

- Statistical description
- Can be effected by a microscopic system and the effect can be macroscopically detected

Meso can serve as a microscope to study micro

Brownian particle was the first mesoscopic device in use
Brownian particle was the first mesoscopic device in use

First paper on Quantum Theory of Solid State (Specific heat)
Annalen der Physik, 22, 180, 800 (1907)

First paper on Mesoscopic Physics
Annalen der Physik, 17, 549 (1905)
Finite size quantum physical systems

Atoms
Nuclei
Molecules

}  Quantum Dots
1. Disorder \( (× – \text{impurities}) \)
2. Complex geometry
3. \( e-e \) interactions
Quantum Dot

1. Disorder ($\times - \text{impurities}$)
2. Complex geometry
3. $e-e$ interactions for a while

Realizations:

- Metallic clusters
- Gate determined confinement in 2D gases (e.g. $GaAs/AlGaAs$)
- Carbon nanotubes
- 
1. Disorder (× – impurities)

2. Complex geometry

How to deal with disorder?

• Solve the Shrodinger equation exactly
• Start with plane waves, introduce the mean free path, and . . .

How to take quantum interference into account?
Electrons in nanostructures

Clean systems without boundaries:

• Electrons are characterized by their momenta or quasimomenta
  ⇒ electronic wave functions are plane waves

• Physics is essentially local

• Interaction between electrons is often apparently not important

In mesoscopic systems:

• Due to the scattering of the electrons off disorder (impurities) and/or boundaries the momentum is not a good quantum number

• Response to external perturbation is usually nonlocal

• Interaction between electrons is often crucial
Lesson 1:
Beyond Markov chains:
Anderson Localization
and
Magnetoresistance
Quantum particle in random quenched potential
Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the “impurity band.” These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.
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Anderson Model

• Lattice - tight binding model
• Onsite energies $\varepsilon_i$ - random
• Hopping matrix elements $I_{ij}$

$-W < \varepsilon_i < W$
uniformly distributed

$I_{ij} = \begin{cases} I & \text{i and j are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$

Anderson Transition

$I < I_c$
Insulator
All eigenstates are localized
Localization length $\xi$

$I > I_c$
Metal
There appear states extended all over the whole system
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhas Pradhan and S. Sridhar

Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

\[ f = 3.04 \text{ GHz} \]  \hspace{1cm} \[ f = 7.33 \text{ GHz} \]

Anderson Insulator \hspace{1cm} Anderson Metal
Classical particle in a random potential

Diffusion

1 particle - random walk

Density of the particles \( \rho \)

Density fluctuations \( \rho(r,t) \) at a given point in space \( r \) and time \( t \).

\[
\frac{\partial \rho}{\partial t} - D \nabla^2 \rho = 0
\]

Diffusion Equation

\( D \) - Diffusion constant

\[
D = \frac{l^2}{d \tau}
\]

\( l \) mean free path

\( \tau \) mean free time

\( d \) # of dimensions
Conductance

\[ G = \sigma L^{d-2} \]

for a cubic sample of the size \( L \)

\[ \sigma = e^2 D \nu \]

\[ \nu \equiv \frac{dn}{d\mu} \]

Einstein - Sutherland Relation for electric conductivity \( \sigma \)

\[ G = \frac{e^2}{h} \left( \nu L^d \right) \frac{Dh}{L^2} \]

\[ g(L) = \frac{hD/\nu L^2}{1/\nu L^d} \]

\[ = \frac{\text{Thouless energy}}{\text{mean level spacing}} \]

Dimensionless Thouless conductance
1. **Mean level spacing**

\[ \delta_1 = \frac{1}{\nu} \times L^d \]

$L$ is the system size; $d$ is the number of dimensions.

2. **Thouless energy**

\[ E_T = \frac{hD}{L^2} \]

$D$ is the diffusion constant.

$E_T$ has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems).

\[ g = \frac{E_T}{\delta_1} \]

dimensionless Thouless conductance

\[ g = \frac{Gh}{e^2} \]
Scaling theory of Localization
(Abrahams, Anderson, Licciardello and Ramakrishnan 1979)

\[ g = \frac{E_T}{\delta_1} \]  

Dimensionless Thouless conductance  
\[ g = \frac{Gh}{e^2} \]

\[ L = 2L = 4L = 8L \ldots \]

without quantum corrections

\[ E_T \propto L^{-2} \quad \delta_1 \propto L^{-d} \]

\[ \frac{d(\log g)}{d(\log L)} = \beta(g) \]
\[
\frac{d(\log g)}{d(\log L)} = \beta(g)
\]

Universal, i.e., material independent

But

It depends on the global symmetries, e.g., it is different with and without $T$-invariance (in orthogonal and unitary ensembles)

\[\beta - \text{function is}\]

\textbf{Limits:}

\begin{align*}
g \gg 1 & \quad g \propto L^{d-2} & \beta(g) &= (d-2) + O\left(\frac{1}{g}\right) \\
g \ll 1 & \quad g \propto e^{-L/\xi} & \beta(g) &\approx \log g < 0
\end{align*}

\begin{array}{c|c|c}
> 0 & d > 2 \\
?? & d = 2 \\
< 0 & d < 2
\end{array}
The $\beta$-function is given by:

$$\frac{d \log g}{d \log L} = \beta(g)$$

The graph shows the $\beta(g)$ function for different dimensions $d$.

- For $d=1$ and $d=2$, all states are unstable.
- For $d=3$, there is a metal-insulator transition at $g_c \approx 1$.

Metal – insulator transition in $3D$

All states are localized for $d=1,2$.
Questions:

• the scaling theory is correct?
• the corrections of the diffusion constant and conductance are negative?

Why diffusion description fails at large scales?
Diffusion description fails at large scales

Why?

Einstein: there is no diffusion at too short scales – there is memory, i.e., the process is not marcovian.
Andrei Markov
1856-1922

Diffusion description fails at large scales
Why?

Einstein: there is no diffusion at too short scales – there is memory, i.e., the process is not marcovian.

Why there is memory at large distances in quantum case?

Quantum corrections at large Thouless conductance – weak localization
Universal description
\[ \frac{d \log g}{d \log L} = \beta(g) \]

\[ \beta(g) = d - 2 + \frac{c_d}{g} \]

\[ g(L) = \sigma_{cl} L^{d-2} - \frac{c_d}{d-2} \quad d \neq 2 \]

\[ \quad - c_2 \log \left( \frac{L}{l} \right) \quad d = 2 \]