Large deviation theory applied to study rare and extreme events in turbulence, atmosphere, and climate dynamics - Lecture II

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Outline

I) Introduction to large deviation theory and its applications to dynamical problems (Wednesday)

II) Large deviation theory for kinetic theories, geostrophic turbulence, and atmosphere dynamics (Thursday)

III) Rare and extreme events in climate dynamics: sampling using rare event algorithms and machine learning (Friday)

Outline for Lecture II.

1. Large deviations for dilute gases (for the Boltzmann equation)
   - Dilute gases and heuristic derivation of the Boltzmann equation
   - Derivation of the large deviation action for dilute gas dynamics
   - The irreversibility paradox
2. Kinetic theory for two dimensional turbulent flows
   - The barotropic quasi-geostrophic model and averaging
   - Kinetic theory of the quasi-geostrophic model
   - An explicit formula for the Reynolds stress for small scale forces
3. Rare transitions and Jupiter's abrupt climate changes
   - Rare transitions for zonal jets
   - Large deviations in the weak noise regime
   - Rare event algorithms and rare transitions for turbulent flows

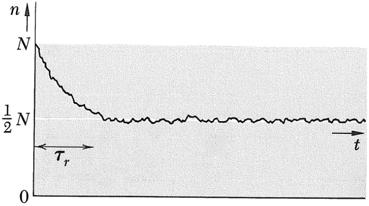
Outline

Boltzmann's Equation

$$\frac{\partial f}{\partial t}+\mathbf{v}.\frac{\partial f}{\partial \mathbf{r}}=\int \mathrm{d}\mathbf{v}_2\mathrm{d}\mathbf{v}_1'\mathrm{d}\mathbf{v}_2'\,w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v},\mathbf{v}_2)\left[f\left(\mathbf{v}'_1,\mathbf{r}\right)f\left(\mathbf{v}_2',\mathbf{r}\right)-f\left(\mathbf{v},\mathbf{r}\right)f\left(\mathbf{v}_2,\mathbf{r}\right)\right].$$

- A cornerstone of physics.
- The irreversibility paradox and the 19th century controversy (Loschmidt, Zermelo, Poincaré).
- Classical explanation of the paradox: Lanford work (1973).
- It is a very active contemporary subject both in physics and mathematics.

Motivation 1: Joule Expansion and Large Deviations



(Figures: Reif)

- What is the probability of a dynamical rare fluctuation? The answer is not known.

4 Dimensional Parameters

- Four dimensional independent parameters: the volume V, the particle number N (or $\rho = N/V$), the energy E (or $\beta = 1/k_B T$ with $E = 3Nk_B T/2$), and the typical value of the cross section a^2.
- A typical velocity is $v_T = \sqrt{1/\beta}$. The mean free path l is determined by
$$\rho a^2 l = 1.$$
A typical collision time is $\tau_c = l/v_T$.
- Orders of magnitude: Hydrogen at the room temperature and standard pressure: $a \simeq 1.4\,10^{-10}$m, $\rho \simeq 2.\,10^{25}\,\mathrm{m}^{-3}$ $(1/\rho^{1/3} = 3.7\,10^{-9})$, $l = 2.5\,10^{-6}$m and $V = 1\,\mathrm{m}^{-3}$, $v_T = 1.6\,10^3\,\mathrm{m.s}^{-1}$, and $\tau_c = 6.7\,10^{-11}$s.

2 Non-Dimensional Parameters

- 4 dimensional independent parameters. We can choose time units and space units.
- We have thus 2 non-dimensional parameters. We choose the number of particles N and the inverse of the number of particle per volume of size l

$$\varepsilon = \left(\rho l^3\right)^{-1} = a^2/l^2 = a^6 n^2$$

- Then $N\varepsilon = V/l^3$.

The Boltzmann–Grad Limit

- The Boltzmann–Grad limit:

$$\varepsilon \to 0 \text{ with either } \varepsilon N \simeq 1 \text{ or } \varepsilon N \gg 1$$

- In this limit, we actually have

$$a \ll \frac{1}{\rho^{1/3}} \ll l,$$

and

$$l \simeq V^{1/3} \text{ or } l \ll V^{1/3}$$

Collision Rates

- A thread of particles with velocities $\mathbf{v}_1$ meets a thread of particle with velocities $\mathbf{v}_2$. We assume homogeneous densities (point Poisson processes).
- Collisions: particle pairs with velocities $(\mathbf{v}_1, \mathbf{v}_2)$ undergo a random change towards particle pairs with velocities $(\mathbf{v}_1', \mathbf{v}_2')$, up to $(\mathrm{d}\mathbf{v}_1', \mathrm{d}\mathbf{v}_2')$.
- This occurs at a rate (in units $\mathrm{m}^{-3}\mathrm{s}^{-1}$) proportional to the $\mathbf{v}_1$ particle density, the $\mathbf{v}_2$ particle density, $\mathrm{d}\mathbf{v}_1'$, and $\mathrm{d}\mathbf{v}_2'$. The proportionality coefficient is called the collision kernel and is denoted

$$\frac{1}{2} w\left(\mathbf{v}_1', \mathbf{v}_2'; \mathbf{v}_1, \mathbf{v}_2\right).$$

- As $w\left(\mathbf{v}_1', \mathbf{v}_2'; \mathbf{v}_1, \mathbf{v}_2\right) \mathrm{d}\mathbf{v}_2' \mathrm{d}\mathbf{v}_1' \rho(\mathbf{v}_1) \rho(\mathbf{v}_2)$ is in units $\mathrm{m}^{-3}\mathrm{s}^{-1}$, w is in units $\mathrm{m}^{-3}\mathrm{s}^{5}$.

Collision Rate Symmetries

- Time reversal symmetry:

$$w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2)=w(-\mathbf{v}_1,-\mathbf{v}_2;-\mathbf{v}_1',-\mathbf{v}_2').$$

- The space rotation symmetry: For $\mathbf{R}$ that belongs to the orthogonal group $SO(3)$

$$w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2)=w(\mathbf{R}\mathbf{v}_1,\mathbf{R}\mathbf{v}_2;\mathbf{R}\mathbf{v}_1',\mathbf{R}\mathbf{v}_2').$$

- Inversion symmetry: The combination of the time reversal symmetry and of the space rotation symmetry for $\mathbf{R}=-\mathbf{I}$, where $\mathbf{I}$ is the identity rotation, gives

$$w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2)=w(\mathbf{v}_1,\mathbf{v}_2;\mathbf{v}_1',\mathbf{v}_2').$$

The Diffusion Cross Section

- The conservation of the momentum and kinetic energy implies

$$w(\mathbf{v}'_1,\mathbf{v}'_2;\mathbf{v}_1,\mathbf{v}_2)=\sigma(\mathbf{v}'_1,\mathbf{v}'_2;\mathbf{v}_1,\mathbf{v}_2)\delta\left(\mathbf{v}_1+\mathbf{v}_2-\mathbf{v}'_1-\mathbf{v}'_2\right)\delta\left(\mathbf{v}_1^2+\mathbf{v}_2^2-\mathbf{v}'^2_1-\mathbf{v}'^2_2\right).$$

- σ, in units of m^2 , is the diffusion cross section.

The Distribution Function

- N particles. Each particle $1 \leq n \leq N$ has a position $\mathbf{r}_n(t)$ and a velocity $\mathbf{v}_n(t)$. We define the empirical distribution as

$$f(\mathbf{r},\mathbf{v},t) \equiv \sum_{n=1}^{N} \delta\left(\mathbf{v}-\mathbf{v}_n(t)\right)\delta\left(\mathbf{r}-\mathbf{r}_n(t)\right).$$

- We assume that the particles evolve according to their own velocity and their mutual collisions only. The evolution of the empirical density is given by

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{r}} = \mathscr{C}$$

where $\mathscr{C}$ accounts for the collision effects.

A Reformulation of Boltzmann's Assumptions

i) The collision duration is neglected compared to the average collision time. The collision geometry is also neglected (point particle assumption).
ii) The probability of three particle encounters is neglected.
iii) ***Propagation of chaos hypothesis*** (Boltzmann's stosszahlansatz)*:* At any time, the effect of $(\mathbf{v}_1,\mathbf{v}_2) \to (\mathbf{v}_1',\mathbf{v}_2')$ collisions on the distribution f can be quantified as if, locally in position space, the particles with velocity $\mathbf{v}_1$ up to $\mathrm{d}\mathbf{v}_1$ and $\mathbf{v}_2$ up to $\mathrm{d}\mathbf{v}_2$ would be mutually statistically independent and each distributed according to a local Poisson point process in position space with densities $\rho(\mathbf{v}_1) = f(\mathbf{r},\mathbf{v}_1,t)\mathrm{d}\mathbf{v}_1$ and $\rho(\mathbf{v}_2) = f(\mathbf{r},\mathbf{v}_1,t)\mathrm{d}\mathbf{v}_2$ respectively.
iv) **Law of large numbers:** The average collision number is evaluated. Possible fluctuations of this number are not considered.

Boltzmann's Equation

- From Boltzmann's hypothesis, we get Boltzmann's equation

$$\frac{\partial f}{\partial t}+\mathbf{v}.\frac{\partial f}{\partial \mathbf{r}}=\int \mathrm{d}\mathbf{v}_2\mathrm{d}\mathbf{v}_1'\mathrm{d}\mathbf{v}_2'\, w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v},\mathbf{v}_2)\left[f\left(\mathbf{v}'_1,\mathbf{r}\right)f\left(\mathbf{v}_2',\mathbf{r}\right)-f\left(\mathbf{v},\mathbf{r}\right)f\left(\mathbf{v}_2,\mathbf{r}\right)\right].$$

- It conserves the mass M, total momentum $\mathbf{P}$ and kinetic energy E.
- The entropy

$$S[f]=-\int \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{r}\, f\log(f)$$

increases along solution to Boltzmann's equation.

Maxwell–Boltzmann Distributions

- Maxwell–Boltzmann distributions:

$$\begin{aligned} f &= \arg\sup \left\{ S \middle| M[f] = N, \mathbf{P}[f] = N\mathbf{p}_0, \text{ and } E[f] = Ne_0 \right\} \\ &= \rho \left(\frac{\beta}{2\pi}\right)^{3/2} \exp\left(-\beta \frac{(\mathbf{v}-\mathbf{p_0})^2}{2}\right). \end{aligned}$$

- Under generic hypothesis on the collision rates w, the equality

$$\frac{\mathrm{d}S}{\mathrm{d}t}[f] \geq 0$$

is strict except for the Maxwell–Boltzmann distributions.

- Then solutions to Boltzmann's equation converge to the Maxwell-Boltzmann distributions.

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Path Large Deviations for the Boltzmann Equation

- Dynamical large deviations for the empirical distribution:

$$P\left[\{f_N(t)\}_{0\leq t<T}=\{f(t)\}_{0\leq t<T}\right]\underset{\varepsilon\downarrow 0}{\asymp}\exp\left(-\frac{\sup_p\int_0^T \mathrm{d}t\left\{\int \dot{f}p\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{v}-H_B\left[f,p\right]\right\}}{\varepsilon}\right).$$

  ε is the inverse of the number of particles in a volume of size the mean free path.

- The large deviation Hamiltonian is $H_B=H_C+H_T$, with H_T the free transport part, and with the collision part H_c given by

$$H_C\left[f,p\right]=\frac{1}{2}\int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{v}_{1,2,1',2'}\,w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2)f(\mathbf{r},\mathbf{v}_1)f(\mathbf{r},\mathbf{v}_2)\left\{e^{\left[p(\mathbf{r},\mathbf{v}_1)+p(\mathbf{r},\mathbf{v}_2)-p(\mathbf{r},\mathbf{v}_1')-p(\mathbf{r},\mathbf{v}_2')\right]}-1\right\}.$$

- C. Leonard, 1995. F. Rezakhanlou, 1998: stochastic model with Boltzmann like behavior.
- F. Bouchet, 2020, for dilute gases.
- T. Bodineau, I. Gallagher, L. Saint-Raymond and S. Simonella, 2020, for a mathematical proof for short times.
- D. Heydecker, 2022, and G. Basile, D. Benedetto, L. Bertini and E. Caglioti, 2022: energy non-conserving solutions with probability $\mathscr{O}\left(e^{-N}\right)$.

A Reformulation of Boltzmann's Assumptions

i) The collision duration is neglected compared to the average collision time. The collision geometry is also neglected (point particle assumption).
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iii) ***Hypothesis of propagation of chaos*** (Boltzmann's stosszahlansatz)*:* At any time, the effect of $(\mathbf{v}_1,\mathbf{v}_2)\to(\mathbf{v}_1',\mathbf{v}_2')$ collisions on the distribution f can be quantified as if, locally in position space, the particles with velocity $\mathbf{v}_1$ up to $\mathrm{d}\mathbf{v}_1$ and $\mathbf{v}_2$ up to $\mathrm{d}\mathbf{v}_2$ would be mutually statistically independent and each distributed according to a local Poisson point process in position space with densities $\rho(\mathbf{v}_1)=f(\mathbf{r},\mathbf{v}_1,t)\mathrm{d}\mathbf{v}_1$ and $\rho(\mathbf{v}_2)=f(\mathbf{r},\mathbf{v}_1,t)\mathrm{d}\mathbf{v}_2$ respectively.
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A Reformulation of Boltzmann's Assumptions

Except for the law of large numbers

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- What is the large deviation rate?

Remember Example 2: Network of Reacting Particles

- N particles X_n, with $1 \leq n \leq N$. Each of them can be in K different states: $X_n = k$ with $1 \leq k \leq K$
- The number of particles which are in the state k is NF_k:

$$\text{Empirical distribution } : \mathbf{F}^N = \{F_k\}_{1 \leq k \leq K}$$

- We assume that the particles have reactions:

$$k_1 + k_2 \to k_3 + k_4 \text{ at a rate } \frac{\lambda_{k_1 k_2 k_3 k_4}}{N}.$$

- Number of reactions per unit of time: of order N. Number of reactions per particle per unit of time: of order 1.
- During a time of order 1, the empirical distribution changes N times, each time with a modification of order $1/N$.
- We expect

$$P\left[\left\{\mathbf{F}^N(t)\right\}_{0 \leq t \leq T} = \{\mathbf{f}(t)\}\right] \underset{N\uparrow\infty}{\asymp} \exp\left(-N\int_0^T \mathrm{d}t\, L\left(\mathbf{f},\dot{\mathbf{f}}\right)\right).$$

Larges Deviations for the Network of Reacting Particles

- The infinitesimal generator for the empirical distribution $\mathbf{F}^N=\{F_k\}_{1\leq k\leq K}$ is

$$G(\mathbf{f})=N\sum_{k_1,k_2,k_3,k_4}\lambda_{k_1k_2k_3k_4}f_{k_1}f_{k_2}\left[\phi\left(f_{k_1}-\frac{1}{N},f_{k_2}-\frac{1}{N},f_{k_3}+\frac{1}{N},f_{k_4}+\frac{1}{N},\tilde{\mathbf{f}}\right)-\phi\left(\mathbf{f}\right)\right].$$

- Using the formula for the Hamiltonian $H(x,p)=\lim_{\varepsilon\downarrow 0}\varepsilon G_{\varepsilon}\left[\mathrm{e}^{\frac{p.x}{\varepsilon}}\right]\mathrm{e}^{-\frac{p.x}{\varepsilon}}$, we get

$$H\left(\mathbf{f},\mathbf{p}\right)=\sum_{k_1,k_2,k_3,k_4}\lambda_{k_1k_2k_3k_4}f_{k_1}f_{k_2}\left(\mathrm{e}^{p_3+p_4-p_2-p_1}-1\right).$$

- Shouldn't the Boltzmann large deviation Hamiltonian be analogous?

Rescaled Distribution Function and Collision Rates

- In units of the mean free path l and of the typical collision time τ_c, velocities are of order one.
- We have $\rho = \varepsilon / l^3$.
- We work with a rescaled distribution function

$$f(\mathbf{r},\mathbf{v},t) \equiv \varepsilon \sum_{n=1}^{N} \delta\left(\mathbf{v}-\mathbf{v}_n(t)\right)\delta\left(\mathbf{r}-\mathbf{r}_n(t)\right).$$

  such that it is of order one in the Boltzmann–Grad limit $\varepsilon \to 0$.
- The cross section σ is of order $a^2 = \varepsilon l^2$. We use a rescaled cross section $\sigma_0 = \sigma / \varepsilon$. The rate is then

$$w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2) = \varepsilon \sigma_0(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2)\delta\left(\mathbf{v}_1+\mathbf{v}_2-\mathbf{v}_1'-\mathbf{v}_2'\right)\delta\left(\mathbf{v}_1^2+\mathbf{v}_2^2-\mathbf{v'}_1^2-\mathbf{v'}_2^2\right).$$

- Each collision occurs at a rate of order ε, produces a change on f of order of ε. On a time scale of order 1, an order one evolution of f results from $1/\varepsilon$ independent collisions. This is clearly a large deviation scaling with rate ε.

Large Deviations Produced by a Large Number of Small Amplitude Independent Moves

- Continuous time Markov processes $\{X_\varepsilon(t)\}_{0\leq t<\infty}$, with infinitesimal generator G_ε.
- We assume that for all $p\in\mathbb{R}^n$ the limit

$$H(x,p)=\lim_{\varepsilon\downarrow 0}\varepsilon G_\varepsilon\left[\mathrm{e}^{\frac{p.x}{\varepsilon}}\right]\mathrm{e}^{-\frac{p.x}{\varepsilon}}.$$

- Then the family X_ε verifies a large deviation principle

$$P\left[\{X_\varepsilon(t)\}_{0\leq t<T}=\{X(t)\}_{0\leq t<T}\right]\underset{\varepsilon\downarrow 0}{\asymp}\exp\left(-\frac{\int_0^T\mathrm{d}t\,L\left(X,\dot{X}\right)}{\varepsilon}\right),$$

with $L(x,\dot{x})=\sup_p\{p\dot{x}-H(x,p)\}$.

(see for instance physics literature, or J. Feng papers, or Freidlin-Wentzell, 3rd edition, 2012)

The Infinitesimal Generator

- We get

$$G[\phi][f] = G_C[\phi][f] + G_T[\phi][f]$$

with

$$G_C[\phi][f] = \frac{1}{2\varepsilon}\int \mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}_2\mathrm{d}\mathbf{v}_1'\mathrm{d}\mathbf{v}_2'\mathrm{d}\mathbf{r}\, w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2) f(\mathbf{r},\mathbf{v}_1) f(\mathbf{r},\mathbf{v}_2)\ldots$$

$$\ldots\left\{\phi\left[f(.)+\varepsilon\left(-\delta_{\mathbf{v}_1}\delta_{\mathbf{r}}-\delta_{\mathbf{v}_2}\delta_{\mathbf{r}}+\delta_{\mathbf{v}_2'}\delta_{\mathbf{r}}+\delta_{\mathbf{v}_1'}\delta_{\mathbf{r}}\right)\right]-\phi[f]\right\}$$

and

$$G_T[\phi][f] = -\int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{v}\,\mathbf{v}.\frac{\partial f}{\partial \mathbf{r}}(\mathbf{r},\mathbf{v})\frac{\delta\phi}{\delta f(\mathbf{r},\mathbf{v})}.$$

Large Deviation Rate Function from the Infinitesimal Generator

- Continuous time Markov processes $\{X_\varepsilon(t)\}_{0\leq t<\infty}$. For instance $X_\varepsilon(t)\in\mathbb{R}^n$.
- We assume that for all $p\in\mathbb{R}^n$ the limit

$$H(x,p)=\lim_{\varepsilon\downarrow 0}\varepsilon G_\varepsilon\left[\mathrm{e}^{\frac{p.x}{\varepsilon}}\right]\mathrm{e}^{-\frac{p.x}{\varepsilon}}$$

- Then the family X_ε verifies a large deviation principle

$$P\left[\{X_\varepsilon(t)\}_{0\leq t<T}=\{X(t)\}_{0\leq t<T}\right]\underset{\varepsilon\downarrow 0}{\asymp}\exp\left(-\frac{\int_0^T \mathrm{d}t\, L\left(X,\dot{X}\right)}{\varepsilon}\right).$$

with rate ε and rate function

$$L(x,\dot{x})=\sup_p\{p\dot{x}-H(x,p)\}.$$

The Action Hamiltonian

- We obtain a large deviation principle for the empirical distribution dynamics

$$P\left[\{f_\varepsilon(t)\}_{0\leq t<T}=\{f(t)\}_{0\leq t<T}\right] \underset{\varepsilon\downarrow 0}{\asymp} \exp\left(-\frac{\sup_p \int_0^T \mathrm{d}t\,\left\{\int \dot{f}\,p\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{v}-H[f,p]\right\}}{\varepsilon}\right).$$

- We get

$$H[f,p]=H_C[f,p]+H_T[f,p] \quad \text{with}$$

$$H_C[f,p]=\frac{1}{2}\int \mathrm{d}\mathbf{v}_{1,2,1',2'}\mathrm{d}\mathbf{r}\, w(\mathbf{v}_1',\mathbf{v}_2';\mathbf{v}_1,\mathbf{v}_2)f(\mathbf{r},\mathbf{v}_1)f(\mathbf{r},\mathbf{v}_2)\left\{e^{\left[p(\mathbf{r},\mathbf{v}_1)+p(\mathbf{r},\mathbf{v}_2)-p(\mathbf{r},\mathbf{v}_1')-p(\mathbf{r},\mathbf{v}_2')\right]}-1\right\}$$

$$\text{and}\quad H_T[f,p]=-\int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{v}\, p(\mathbf{r},\mathbf{v})\mathbf{v}.\frac{\partial f}{\partial \mathbf{r}}(\mathbf{r},\mathbf{v}).$$

Boltzmann's Equation is the Most Probable Evolution

- The corresponding fluctuation paths (most probable paths) solve $\frac{\partial f}{\partial t} = \frac{\delta H}{\delta p}[f,0]$.
- It is easily checked that it is Boltzmann's equation

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{r}} = \int \mathrm{d}\mathbf{v}_2 \mathrm{d}\mathbf{v}_1' \mathrm{d}\mathbf{v}_2'\, w(\mathbf{v}_1', \mathbf{v}_2'; \mathbf{v}, \mathbf{v}_2) \left[f\left(\mathbf{v}'_1, \mathbf{r}\right) f\left(\mathbf{v}_2', \mathbf{r}\right) - f\left(\mathbf{v}, \mathbf{r}\right) f\left(\mathbf{v}_2, \mathbf{r}\right) \right].$$

Conserved Quantities

- If C is equal to either the mass

$$M = \int d\mathbf{r}d\mathbf{v}\, f,$$

or the momentum

$$\mathbf{P} = \int d\mathbf{r}d\mathbf{v}\, \mathbf{v} f,$$

or the kinetic energy

$$E = \frac{1}{2}\int d\mathbf{r}d\mathbf{v}\, \mathbf{v}^2 f,$$

we easily check that

$$\text{for any } f \text{ and } p,\ \int d\mathbf{r}d\mathbf{v}\, \frac{\delta H}{\delta p(\mathbf{r},\mathbf{v})} \frac{\delta C}{\delta f(\mathbf{r},\mathbf{v})} = 0.$$

- As expected, the mass, momentum, and kinetic energy are conserved quantities.

Is Entropy the Quasipotential

- We expect from equilibrium statistical, that for dilute gazes (non interacting particle limit)

$$U[f]=\begin{cases} -S[f] \text{ if } M[f]=N\varepsilon=V/l^3,\ \mathbf{P}[f]=N\mathbf{p}_0,\ \text{and } E[f]=Ne_0 \\ -\infty \text{ otherwise.} \end{cases}$$

- Can we check this directly from our action?

Outline

Definition of Detailed Balance for Stochastic Processes

- We consider a time homogeneous stationary stochastic process $\{X(t)\}_{0\leq t<\infty}$ (for instance a continuous time Markov process).
- P_S is the stationary probability distribution function, and P is the two point transition probability distribution function

$$P_S(x)=\mathbb{E}\left[\delta\left(x-X(t)\right)\right] \text{ and } P\left(y,T;x,0\right)=\mathbb{E}_x\left[\delta\left(y-X\left(T\right)\right)\right].$$

- The definition of time reversibility for this process is

$$\text{for any } \left(x,y,T\right),\; P\left(y,T;x,0\right)P_S(x)=P\left(x,T;y,0\right)P_S(y).$$

This is called the detailed balance condition.

- If the N-particle dynamics is time-reversible (for instance Hamiltonian), we expect the stochastic process of the empirical distribution to be time reversible. How does this translate at the level of the path large deviations?

Detailed Balance Condition for Large Deviations

- **Detailed balance condition** for path large deviations:

$$\text{for any x and } \dot{x},\ L(x,\dot{x})-L(x,-\dot{x})=\dot{x}.\nabla U,$$

  or equivalently

$$\text{for any x and p},\ H(x,-p)=H(x,p+\nabla U).$$

- I is the time-reversal symmetry involution. We assume that I is self adjoint for the scalar product $I(x).p=I(p).x$. ***Generalized detailed balance*** condition: if $U(x)=U(I[x])$ and

$$L(x,\dot{x})-L(x,-I[\dot{x}])=I[\dot{x}].\nabla U$$

  or equivalently

$$H(I[x],-I[p])=H(x,p+\nabla U).$$

- All the large deviation Hamiltonians for kinetic theories verify this large deviation detailed balance condition.

Dynamical Large Deviations and the Irreversibility Paradox of Kinetic Theories

- We expect a large deviation action which is time-reversal symmetric with respect to the entropy.
- The time reversal symmetry is not broken neither by the mesoscopic description nor by the Stosszahlansatz!
- However the most-probable evolution (or the average, due to the law of large number) is irreversible. It increase entropy.
- Fluctuation paths are time reversed relaxation paths **(non-linear Onsager relations)**.
- The picture is clear and simple. There is no more any paradox. Any path is possible. The probability of any path is quantified.

Path large deviations for kinetic theories

$$f_N(\mathbf{r},\mathbf{v},t) \equiv \frac{1}{N}\sum_{n=1}^{N}\delta\left(\mathbf{v}-\mathbf{v}_n(t)\right)\delta\left(\mathbf{r}-\mathbf{r}_n(t)\right).$$

$$\mathbb{P}\left[\{f_N(t)\}_{0\le t<T}=\{f(t)\}_{0\le t<T}\right] \underset{\varepsilon\downarrow 0}{\asymp} \exp\left(-\frac{\sup_p \int_0^T \mathrm{d}t\,\left\{\int \dot{f}\,p\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{v}-H[f,p]\right\}}{\varepsilon}\right).$$

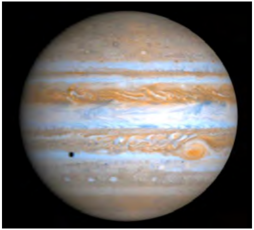
- What is ε? Can we compute H?
- Dilute gases (Boltzmann equation). F. Bouchet, JSP, 2020.
- Plasma beyond debye length. O. Feliachi and F. Bouchet, JSP, 2021.
- Systems with long range interactions. O. Feliachi and F. Bouchet, JSP, 2022.
- Weak turbulence theory (wave turbulence). J. Guioth, G. Eyink and F. Bouchet, 2022, arXiv:2203.11737.

Slides for lectures at ENS-Lyon, SISSA and the Weizmann Institute.

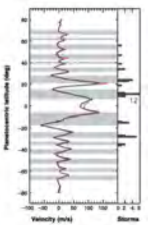
Outline

Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



Jupiter's atmosphere



Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s,$$

where $\omega=(\nabla\wedge\mathbf{v}).\mathbf{e}_z$ is the vorticity, $q=\omega+\beta y$ is the Potential Vorticity (PV), f_s is a random Gaussian field with correlation $\langle f_S(\mathbf{x},t)f_S(\mathbf{x}',t')\rangle=C(\mathbf{x}-\mathbf{x}')\delta(t-t')$, ε is the average energy input rate, λ is the Rayleigh friction coefficient.

- Spin up or spin down time $=1/\alpha\ll 1=$ jet inertial time scale.
- A reasonable model for Jupiter's zonal jets.

The 2D Stochastic Navier-Stokes Equations ($\beta = 0$)

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = \nu \Delta \omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Bricmont, Debussche, Hairer, Kuksin, Kupiainen, Mattingly, Shirikyan, Sinai, ...
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu \to 0}\ \mu_\nu$,
  - In this limit, almost all trajectories are solutions of the 2D Euler equations.

Kuksin, S. B., & Shirikyan, A. (2012). Mathematics of two-dimensional turbulence. Cambridge University Press.

- We would like to describe the invariant measure

Dynamics of the Barotropic Quasi-Geostrophic Equations

Top: Zonally averaged vorticity (Hovmöller diagram and red curve) and velocity (green). Bottom: vorticity field

Which Mathematical Framework for the Inertial Limit?

- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=-\alpha\omega+\sqrt{2\alpha}f_s.$$

- Inertial limit: spin up or spin down time $=1/\alpha \gg 1=$ jet inertial time scale (a relevant assumption for Jupiter).
- This is an averaging problem for an Hamiltonian system perturbed by weak non Hamiltonian forces.
- The Hamiltonian system is an infinite dimensional one with an infinite number of conserved quantities.

Dynamics of the Barotropic Quasi-Geostrophic Equations

Top: Zonally averaged vorticity (Hovmöller diagram and red curve) and velocity (green). Bottom: vorticity field

Decomposition Between Zonal Jets and Turbulence: A Slow/Fast Dynamical System

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = -\alpha\omega + \sqrt{2\alpha} f_s \text{ with } \alpha \ll 1$$

- Time scale separation. We decompose into slow (zonal flows) and fast (eddy turbulence) variables

$$U(y)\mathbf{e}_x = \langle \mathbf{v} \rangle \equiv \frac{1}{2\pi}\int_{\mathscr{D}} \mathrm{d}x\,\mathbf{v} \text{ and } \mathbf{v} = U(y)\mathbf{e}_x + \sqrt{\alpha}\mathbf{v}_m.$$

- Stochastic averaging using the time scale separation ($\tau = \alpha t$).

$$\frac{\partial U}{\partial \tau} = F(U).$$

Averaging for the Stochastic Quasi-Geostrophic Eq.

$$\frac{\partial U}{\partial \tau} = F(U).$$

- $F[U] = -\mathbb{E}_U \langle v_{m,y} q_m \rangle$. The average of the Reynolds stress is over the statistics of the quasilinear inertial dynamics:

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = -\alpha q_m + f_s$$

and

$$\langle v_{m,y} q_m \rangle = \frac{1}{L_y}\int \mathrm{d}y\, v_{m,y} q_m.$$

- We identify SSST by Farrell and Ioannou (JAS, 2003); quasilinear theory by Bouchet (PRE, 2004); CE2 by Marston, Conover and Schneider (JAS, 2008); Sreenivasan and Young (JAS, 2011).

Dynamics of the Relaxation to the Averaged Zonal Flows

The turbulence has been averaged out

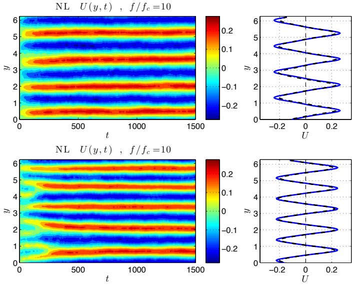
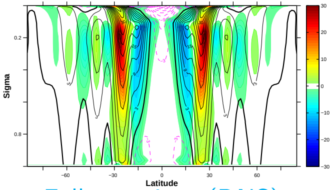


Figure by P. Ioannou (Farrell and Ioannou)

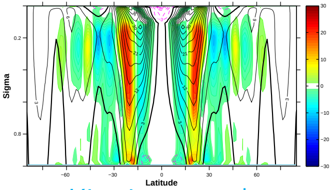
- Extremely efficient numerical simulation of the averaged jet dynamics.

Averaging and Atmosphere Jets

Troposphere Dynamics: comparison of averaging and a direct numerical simulation



Full equations (DNS).



Kinetic approach.

Zonal wind and momentum convergence for the primitive equations.

Farid Ait Chaalal and Tapio Schneider (Caltech and ETH Zurich).

- The qualitative structure of a **fast** rotating Earth troposphere is well approximated by the averaged equations.

The Earth Jet Stream

Higher troposphere winds

Outline

Decomposition Between Zonal Jets and Turbulence: A Slow/Fast Dynamical System

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = -\alpha\omega + \sqrt{2\alpha} f_s \text{ with } \alpha \ll 1$$

- Time scale separation. We decompose into slow (zonal flows) and fast (eddy turbulence) variables

$$U(y)\mathbf{e}_x = \langle \mathbf{v} \rangle \equiv \frac{1}{2\pi} \int_{\mathscr{D}} \mathrm{d}x\, \mathbf{v} \text{ and } \mathbf{v} = U(y)\mathbf{e}_x + \sqrt{\alpha}\mathbf{v}_m.$$

- Stochastic averaging using the time scale separation.

$$\frac{\partial U}{\partial \tau} = F(U).$$

- Mathematics: we have identified a small parameter such that a stochastic averaging approach might be relevant.

Averaging for the Stochastic Quasi-Geostrophic Eq.

$$\frac{\partial U}{\partial \tau} = F(U).$$

- $F[U] = -\mathbb{E}_U \langle v_{m,y} q_m \rangle$. The average of the Reynolds stress is over the statistics of the quasilinear inertial dynamics:

$$\partial_t q_m + U(y) \frac{\partial q_m}{\partial x} + v_{m,y} \frac{\partial q_z}{\partial y} = -\alpha q_m + f_s$$

and

$$\langle v_{m,y} q_m \rangle = \frac{1}{L_y} \int \mathrm{d}y \, v_{m,y} q_m .$$

- Mathematics: questions about the ergodicity of the fast process. Classical results about averaging do not apply.

Proof for the Validity of Averaging and Inviscid Damping of the 2D Euler Equations

- The first step is to understand the asymptotic stability of zonal jets $\mathbf{v} = U(y)\mathbf{e}_x$ for the inertial equation, the 2D Euler equation:

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = 0.$$

- A problem analogous to nonlinear Landau damping for Vlasov equation (Mouhot, and Villani, 2010).
- Inviscid damping for the linearized 2D Euler equations for any stable shear flow $U(y)$ (Bouchet and Morita, 2010).
- Proof of the inviscid relaxation (asymptotic stability) of the velocity field for linear shear flows $U(y) = sy\mathbf{e}_x$ (Bedrossian and Masmoudi, 2015).

Inviscid Damping of the Linearized Euler Eq.

- Base state: a stable steady state $\mathbf{v}_0 = U(y)\mathbf{e}_x$, with vorticity $\Omega(y)$: $\mathbf{v}_0.\nabla Q = 0$

$$\partial_t \omega_m + U(y)\frac{\partial \omega_m}{\partial x} + v_{m,y}\frac{\partial \Omega}{\partial y} = 0 \text{ with } \omega(t=0) = \omega_0 .$$

- For the linearized 2D Euler equation and non-monotonous base flow, the velocity field decreases algebraically at large times

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t}\exp\left(-ikU(y)t\right) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2}\exp\left(-ikU(y)t\right).$$

F. Bouchet and H. Morita, 2010, Physica D.

Validity of Averaging?

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = -\alpha\omega_m + \sqrt{2}f_s$$

- We need to prove that the Gaussian process has an invariant measure which has a limit when $\alpha \to 0$.
- For the linearized 2D Euler equation and non-monotonous base flow, the velocity field decreases algebraically at large times

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t}\exp\left(-ikU(y)t\right) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2}\exp\left(-ikU(y)t\right).$$

Invariant Measure in the Inertial Limit

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = -\alpha\omega_m + \sqrt{2}f_s$$

- The two point correlation function $\mathbb{E}_\alpha\left(\mathbf{v}_m(y_1)\mathbf{v}_m(y_2)\right)$ has a limit when $\alpha \downarrow 0$.
- The two point correlation function $\mathbb{E}_\alpha\left(q_m(y_1)q_m(y_2)\right)$ has a limit when $\alpha \downarrow 0$., as a distribution.
- The two two point correlation function $\mathbb{E}_\alpha\left(\nabla q_m(y_1)\nabla q_m(y_2)\right)$ diverges when $\alpha \downarrow 0$.
- The Reynolds stress $\mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)$ has a limit when $\alpha \downarrow 0$.

F. Bouchet, C. Nardini and T. Tangarife, J. Stat. Phys., 2013

Reynolds Stress Ergodicity in the Inertial Limit

- The Reynolds stress $\mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)$ has a limit when $\alpha\downarrow 0$.
- Pointwise divergence of the ergodic average:

$$\mathbb{E}_\alpha\left\{\left[\frac{1}{T}\int_0^T \mathrm{d}t\, v_{m,y}(y)q_m(y)-\mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)\right]^2\right\}\underset{T\uparrow\infty\ \alpha\downarrow 0}{\sim}\frac{A(y)}{\alpha T}$$

- Convergence as a distribution: for any test function ϕ

$$\mathbb{E}_\alpha\left\{\int \mathrm{d}y\,\phi(y)\left[\frac{1}{T}\int_0^T \mathrm{d}t\, v_{m,y}(y)q_m(y)-\mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)\right]\right\}^2\underset{T\uparrow\infty\ \alpha\downarrow 0}{\sim}\frac{A}{T}$$

F. Bouchet, C. Nardini and T. Tangarife, J. Stat. Phys., 2013

T. Tangarife's PhD thesis, 2015.

Validity of Averaging at the Level of the Law of Large Numbers

- The fast variable dynamics has an invariant measure in the inviscid limit thanks to inviscid damping.
- Velocity like observables have a finite expectation in the inviscid limit.
- The Reynolds stress has a finite expectation in the inviscid limit.
- The ergodic average of the Reynolds stress converges as a distribution.
- We have partial answers only, for the validity of averaging!

F. Bouchet, C. Nardini and T. Tangarife, J. Stat. Phys., 2013

T. Tangarife's PhD thesis, 2015.

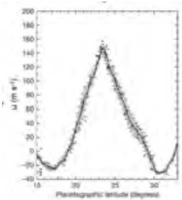
Outline

An Explicit and "Universal" Formula for the Reynolds Stress

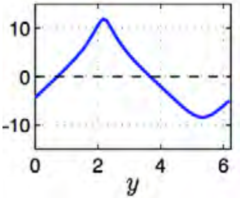
$$\frac{\partial U}{\partial \tau} = F(U)$$

- For small scale forces we have an explicit expression for $F(U)$

$$F(U) = -\frac{\partial}{\partial y}\left(\frac{\varepsilon}{\partial U/\partial y}\right) - \alpha U,$$ where ε is the energy injection rate.



Jupiter's velocity profile
(Sanchez Lavega, 2008)



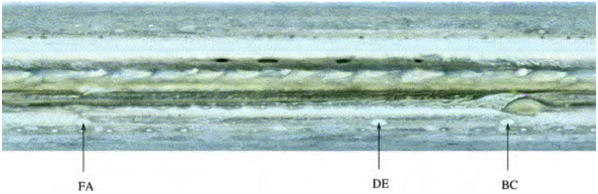
Theoretical/numerical velocity profile

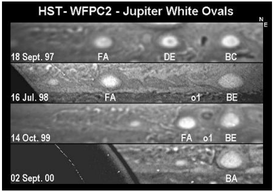
E. Woillez and F. Bouchet, EPL 2017 and JFM 2019

Outline

Jupiter's Abrupt Climate Change

Have we lost one of Jupiter's jets ?

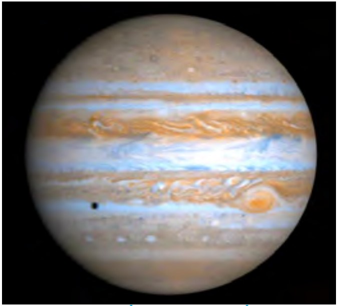




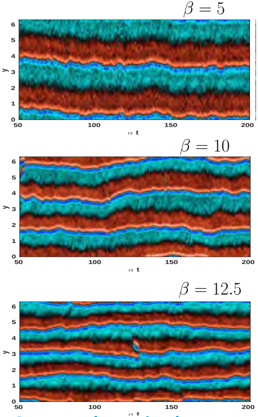
Jupiter's white ovals (see Youssef and Marcus 2005)

The white ovals appeared in 1939-1940 (Rogers 1995). Following an instability of one of the zonal jets?

Multistability for Quasi-Geostrophic Jets



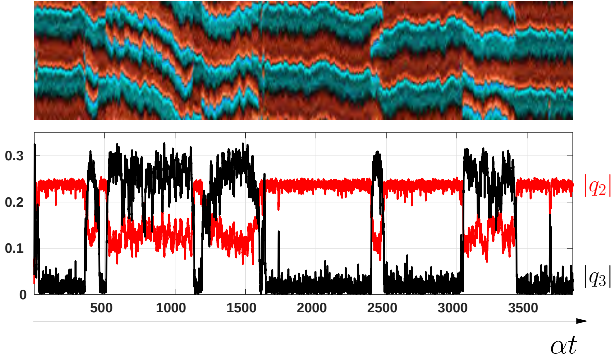
Jupiter's atmosphere



QG zonal turbulent jets

- Multiple attractors had been observed previously by B. Farrell and P. Ioannou.

Rare Transitions Between Quasigeostrophic Jets



Rare transitions for quasigeostrophic jets (with E. Simonnet)

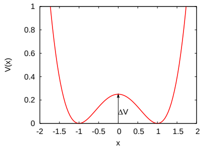
- This is the first observation of spontaneous transitions.
- How to predict those rare transitions? What is their probability? Which theoretical approach?

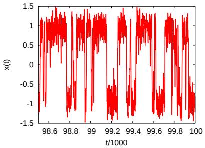
Outline

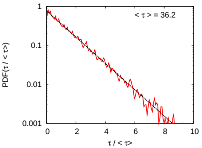
Kramer's Problem: a Pedagogical Example for Bistability

Historical example: Computation by Kramer of Arrhenius' law for a bistable mechanical system with stochastic noise

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T_e}\eta(t) \text{ Rate : } \lambda = \frac{1}{\tau}\exp\left(-\frac{\Delta V}{k_B T_e}\right).$$







The problem was solved by Kramer (30'). Modern approach: path integral formulation (instanton theory, physicists) or large deviation theory (Freidlin–Wentzell, mathematicians).

Freidlin–Wentzell Theory

- For dynamical systems with weak noises

$$\frac{d\mathbf{x}}{dt}=\mathbf{b}(\mathbf{x})+\sqrt{2\varepsilon}\eta\left(t\right).$$

- Path integral representation of transition probabilities (Onsager–Machlup, 53'):

$$P\left(x_{-1},T;x_{1},0\right)=\int_{x(0)=x_{1}}^{x(T)=x_{-1}}\mathrm{e}^{-\frac{1}{4\varepsilon}\int_{0}^{T}\left[\dot{x}-b(x)\right]^{2}}\mathscr{D}\left[x\right].$$

- We consider a saddle point approximation (WKB), and obtain the Arrhenius law as a large deviation result $\lambda \underset{\varepsilon\downarrow 0}{\asymp} \mathrm{e}^{-\frac{\Delta V}{\varepsilon}}$ with

$$\Delta V=\inf_{T\geq 0}\inf_{\left\{x(t)\middle|x(0)=x_{1}\text{ and }x(T)=x_{-1}\right\}}\left\{\frac{1}{4}\int_{0}^{T}\left[\dot{x}-b(x)\right]^{2}\,\mathrm{d}t\right\}$$

Most Transition Paths Follow the Instanton

- In the weak noise limit, most transition paths follow the most probable path (instanton)

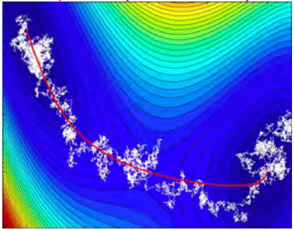


Figure by Eric Van den Eijnden

- Arrhenius law then follows, for both gradient (reversible) and non gradient (irreversible) dynamics

$$\lambda \underset{\varepsilon\to 0}{\asymp} e^{-\frac{\Delta V}{\varepsilon}}.$$

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta y$ is the Potential Vorticity (PV), f_s is a random Gaussian field with correlation $\langle f_S(\mathbf{x},t) f_S(\mathbf{x}',t')\rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t')$, ε is the average energy input rate, λ is the Rayleigh friction coefficient.

- Spin up or spin down time $= 1/\alpha \ll 1 =$ jet inertial time scale.
- Using stochastic averaging we might justify a Freidlin-Wentzel formalism for the slow U field.

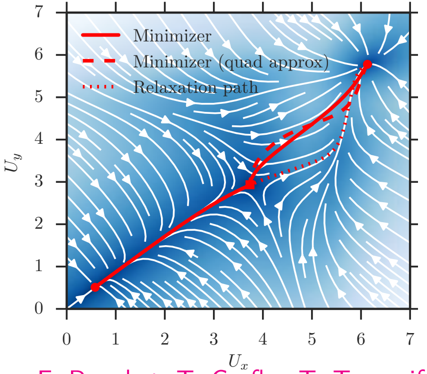
Rare Transitions and Stochastic Averaging for the Quasi-Geostrophic Dynamics

$$\frac{\partial U_\alpha}{\partial t} = F(U_\alpha) + \sqrt{\alpha}\sigma(U_\alpha, t)$$

- $\sigma(U_\alpha)$ quantify the statistics of the time-averaged Reynolds stress fluctuations.
- Using stochastic averaging we might justify a Freidlin-Wentzel formalism for the slow U field.

Gaussian Fluctuations Do Not Describe Rare Transitions

Example of a solvable toy model for $\dfrac{\partial U_\alpha}{\partial t} = F(U_\alpha) + \sqrt{\alpha}\sigma(U_\alpha, t)$:



(Figure from F. Bouchet, T. Grafke, T. Tangarife, and E. Vanden-Eijnden, J. Stat. Phys. 2016)

The Large Deviations that Describe Rare Transitions

$$\begin{cases} \frac{\mathrm{d}U_\alpha}{dt} & = & f(U_\alpha, \mathbf{v}_t) \\ \frac{\mathrm{d}\mathbf{v}_t}{dt} & = & \frac{1}{\alpha} g(U_\alpha, \mathbf{v}_t) + \frac{1}{\sqrt{\alpha}} h \frac{\mathrm{d}W}{\mathrm{d}t} \end{cases}$$

- The equation for the slow variable is $\frac{\partial U_\alpha}{\partial t} = F(U_\alpha) + \sqrt{\alpha}\sigma(U_\alpha, t)$,

$$\text{with } \mathbb{P}(\{U_\alpha(t)\} = \{U(t)\}) \underset{\alpha \downarrow 0}{\asymp} \exp\left\{-\frac{1}{\alpha}\int_0^T \mathrm{d}t\, L\left[U, \dot{U}\right]\right\},$$

$$L[U, \dot{U}] = \sup_P \left[\dot{U}P - H(U,P)\right], \text{ and}$$

$$H(U,P) = \lim_{T\to\infty} \frac{1}{T} \log \mathbb{E}_U \left\{\exp\left[P\int_O^T f(U, \mathbf{v}_t(t))\right]\right\}$$

Freidlin-Wentzell textbook, Veretennikov (2000), Kifer (2007).

For Turbulent Flows: Large Deviations of Quadratic Observables for an Ornstein–Uhlenbeck Process

- For quadratic in $\mathbf{v}_t$ f, and linear g and h (for instance for the quasigeostrophic model), H solves a nonlinear Lyapunov eq.

$$\left\{\begin{array}{lll} \frac{\mathrm{d}U_\alpha}{dt} & = & \mathbf{v}_t^T M_{U_\alpha}\mathbf{v}_t - U_\alpha \\ \frac{\mathrm{d}\mathbf{v}_t}{dt} & = & \frac{1}{\alpha}L_{U_\alpha}\mathbf{v}_t + \frac{1}{\sqrt{\alpha}}h\frac{\mathrm{d}W}{\mathrm{d}t} \end{array}\right.$$

$$H(U,P) = \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}_U\left\{\exp\left[P\int_0^T \mathbf{v}_t^T M_U \mathbf{v}_t\right]\right\} = \mathrm{Tr}(CN_\infty),$$

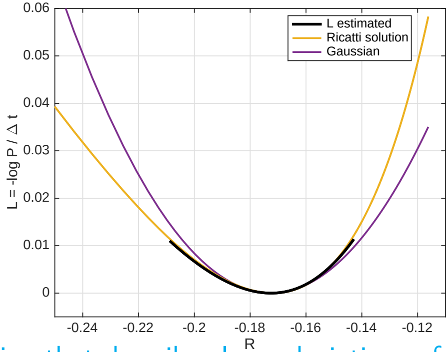
where C is the noise correlation function and N_∞ is the asymptotic solution of the matrix Ricatti equation

$$\frac{\partial N}{\partial t} + L_U^T N + NL_U = 2NCN + PM.$$

We can solve this equation explicitely sometimes, or numerically.

F. B., T. Grafke, T. Tangarife, and E. Vanden-Eijnden, J. Stat. Phys., 2016, T. Tangarife's PhD thesis, and F.B., B.M., and T.T., 2017 POF.

Lagrangian for the Large Deviations of the Time Averaged Reynolds Stress



The Lagrangian that describes large deviations of Reynold stresses (one point statistics)

- In general, rare transitions involve non Gaussian fluctuations.

T. Tangarife's PhD thesis, F.B., B.M., and T.T., 2017 POF.

Work in Progress: Compute the Instantons and the Transition Rates

$$\frac{\partial U_\alpha}{\partial t} = F(U_\alpha) + \sqrt{\alpha}\sigma(U_\alpha, t)$$

- The transition rate λ from an attractor U_0 to an attractor U_F verifies an Arrhenius law

$$\lambda \underset{\alpha\downarrow 0}{\asymp} \mathrm{e}^{-\frac{\Delta V}{\alpha}} \text{ with } \Delta V = \inf_{\{U(t)\left|U(0)=U_0 \text{ and } U(T)=U_F,\right\}} \int_0^T \mathrm{d}t\, L\left[U,\dot{U}\right].$$

- And the instanton is the minimizer.

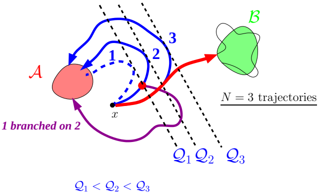
Outline

1. Large deviations for dilute gases (for the Boltzmann equation)
   - Dilute gases and heuristic derivation of the Boltzmann equation
   - Derivation of the large deviation action for dilute gas dynamics
   - The irreversibility paradox
2. Kinetic theory for two dimensional turbulent flows
   - The barotropic quasi-geostrophic model and averaging
   - Kinetic theory of the quasi-geostrophic model
   - An explicit formula for the Reynolds stress for small scale forces
3. Rare transitions and Jupiter's abrupt climate changes
   - Rare transitions for zonal jets
   - Large deviations in the weak noise regime
   - Rare event algorithms and rare transitions for turbulent flows

Rare Events and Adaptive Multilevel Splitting (AMS)

AMS: an algorithm to compute rare events, for instance rare transition paths

- Rare event algorithms: Kahn and Harris (1953), Chandler, Vanden-Eijnden, Schuss, Del Moral, Dupuis, ...
- The adaptive multilevel splitting algorithm:



AMS algorithm

Strategy: selection and cloning.
Probability estimate:

$$\hat{\alpha} = (1 - 1/N)^K, \text{ where}$$

N is the clone number and K the iteration number.

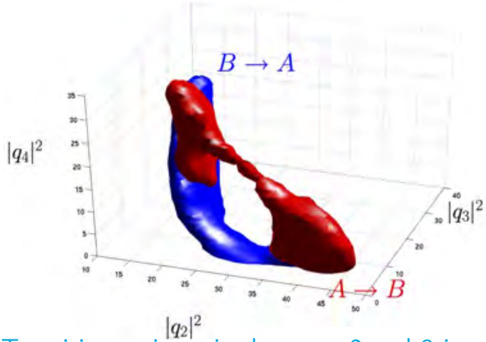
Cérou, Guyader (2007). Cérou, Guyader, Lelièvre, and Pommier (2011).

A Transition from 2 to 3 Jets

Top: Zonally averaged vorticity (Hovmöller diagram and red curve) and velocity (green). Bottom: vorticity field

Atmosphere Jet "Instantons" Computed using the AMS

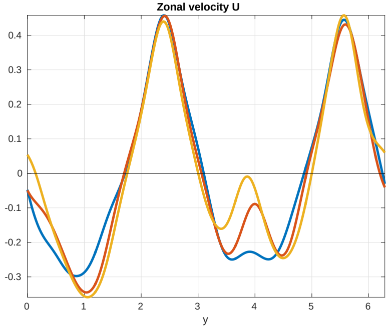
AMS: an algorithm to compute rare events, for instance rare reactive trajectories

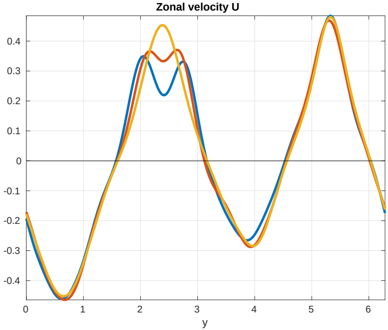


Transition trajectories between 2 and 3 jet states

- The dynamics of turbulent transitions is predictable.
- Asymmetry between forward and backward transitions.

Evolution of Velocity Fields During the Transition





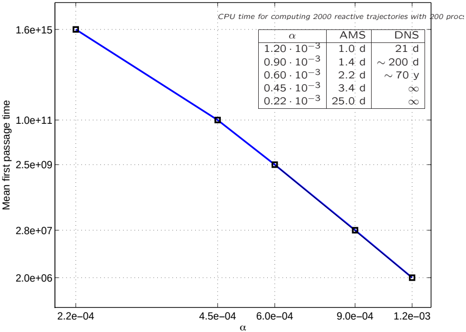
Nucleation of a new jet

Merging of two jets

- Asymmetry between forward and backward transitions.

Transition Rates for Unreachable Regimes Through DNS

With the AMS we can estimate huge average transition times



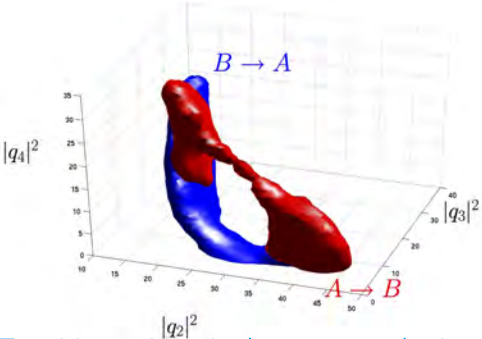
α	AMS	DNS
$1.20 \cdot 10^{-3}$	1.0 d	21 d
$0.90 \cdot 10^{-3}$	1.4 d	$\sim$ 200 d
$0.60 \cdot 10^{-3}$	2.2 d	$\sim$ 70 y
$0.45 \cdot 10^{-3}$	3.4 d	∞
$0.22 \cdot 10^{-3}$	25.0 d	∞

Average transition time versus α

- With the AMS algorithm, we study transitions that would require an astronomical computation time using direct numerical simulations.

Atmosphere Jet "Instantons" Computed using the AMS

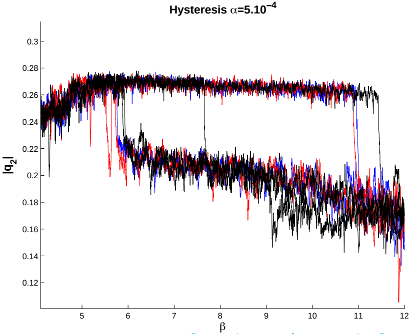
AMS: an algorithm to compute rare events, for instance rare reactive trajectories



Transition trajectories between 2 and 3 jet states

- The dynamics of turbulent transitions is predictible.
- Asymmetry between forward and backward transitions.

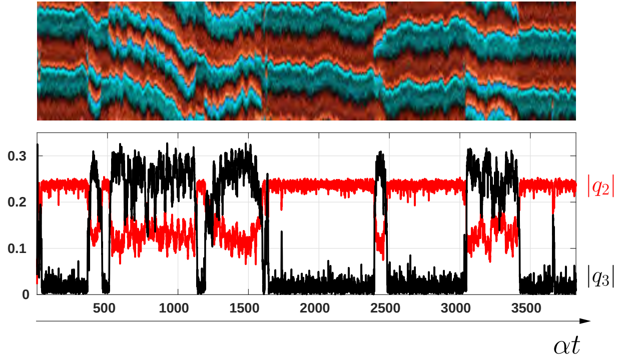
A Complex Internal Dynamics for the 3-Jet States



Hysteresis experiment for the 2/3 jet bifurcations

- The 3 jet states have larger fluctuations than the 2 jet states.

Rare Transitions Between Quasigeostrophic Jets

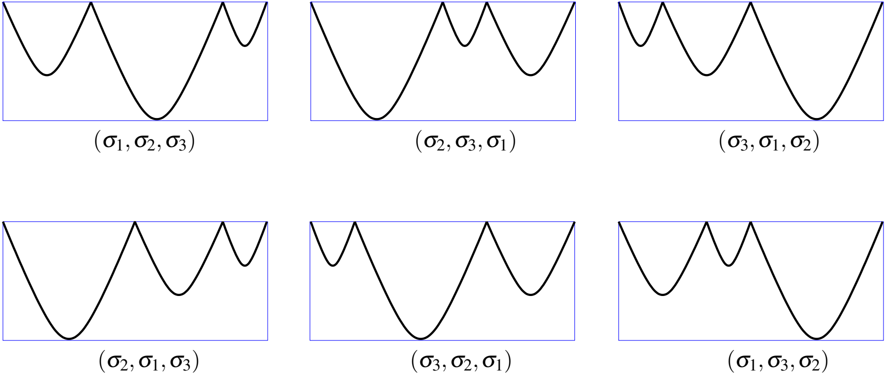


Rare transitions for quasigeostrophic jets

- It seems that the 3 jet states might have different structures.

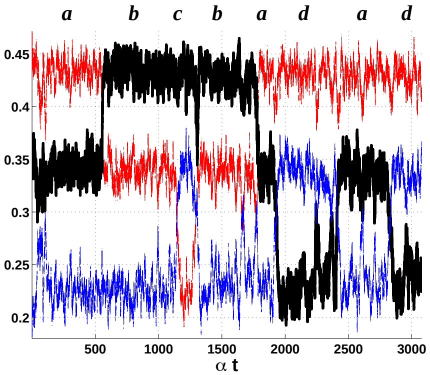
A Family of Different 3-Jet Attractors

Symmetry breaking within the set of 3-jet attractors



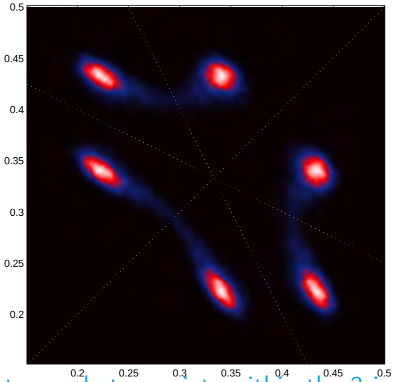
Schematic zonal velocity fields $U(y)$ for the 3-jet attractors

Internal Multistability for the 3-Jet Attractors



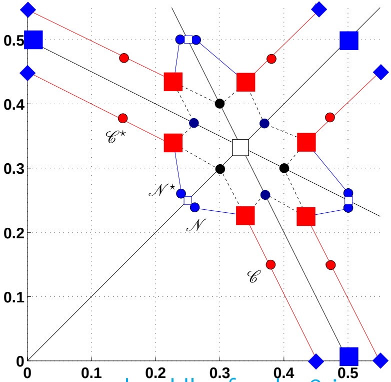
Timeseries for the distance between jets within the 3-jet attractors

Internal Multistability for the 3-Jet Attractors



PDF of distances between jets within the 3-jet attractors.

Bifurcation Diagram for the 3-Jet Attractors



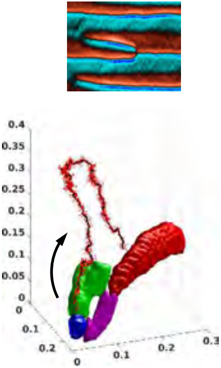
Attractors and saddles for the 3 jet states

Each axe represent one of the 3 distances between the 3 jets.

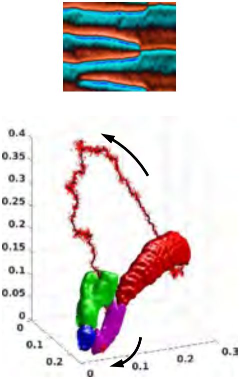
A Richer Transition Phenomenology

Transitions through states with four jets are possible

$T = 4.3$



$T = 2.7$



Conclusions

- We have computed rare transitions between zonal jets, similar to Jupiter's abrupt climate changes, that can not be computed using direct numerical simulations (with E.S.).
- We have partial results for the justification of averaging (ergodicity, etc ...), (with C.N., and T.T.).
- For small scale forces, the average Reynolds stress can be computed explicitly and is universal. We have a good qualitative agreement with Jupiter's jets. (with E.W.).
- The rare transitions involve non-Gaussian fluctuations of the Reynolds stress. (with T.G., B.M., T.T., and E.V-E).

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