Plan of Lectures:

Review Kitaev's result that determining ground-state energy of 5-local Hamiltonians is a hard problem even for quantum computers.

Extend Kitaev's result to geometrically-local systems (1D and 2D lattices)
- contrast with classical spin systems
- use of dynamic programming techniques

Present Bravyi's result, namely that it is easy to decide whether a 2-local (2-body) Hamiltonian on qubits (spin 1/2) is frustration-free.

5-local Hamiltonian problem \((LH) \in BQP ("quantum NP")\).

\[ H = \sum_{i=1}^{5} H_i \quad \text{5-local: } H_i \text{ acts nontrivially on 5 qubits} \]

\[ 0 \leq H_i \leq I, \ n \text{ qubits in total, } H = \text{poly}(n) \]

\[ x \equiv (n, \alpha, a, b) \quad 0 \leq a < b \quad \beta - a = O(n^{-\alpha}) \quad \alpha > 0 \]

\[ f(n) = S \left( g(n) \right) \]

if \( \exists \ c, \ n \text{ s.t.} \]

\[ f(n) \geq c \cdot g(n) \]

(Example 2-local \(H\): AF Heisenberg model on a graph \(G = (V, E)\)

\[ |V| = n \]

\[ \sum_{\text{edges } E_i} s_i \]

\( LH \) Problem: decide whether lowest eigenvalue \( \gamma(H) \leq a \) ("YES")

\[ \gamma(H) > b \] ("No")
Why is LH problem in QMA?

If YES someone can give you groundstate $|\psi_0\rangle$ and you can efficiently estimate $\langle\psi_0 |H| \psi_0\rangle \leq a$ using a quantum circuit.

If NO no one can give you a state $|\psi\rangle$ with $\langle\psi |H| \psi\rangle \leq a$ or... upon getting $|\psi\rangle$, you calculate $\langle\psi |H| \psi\rangle$ using the quantum circuit and you are guaranteed that $\langle\psi |H| \psi\rangle \geq b$.

Why is LH problem "QMA-hard"?

Kitaev: you can map any problem that you can solve in QMA onto instances of the $5$-local LH problem.

So solving any $5$-local LH problem would allow you to solve all problems in QMA.

Kitaev's construction is based on a map between a quantum circuit to a Hamiltonian, or from dynamics to a ground state which is a superposition of states at different times (history state) 'quantize time'.
Quantum circuit $U_1, U_2, \ldots, U_n$ gates\n
Hamiltonian on $n_1 + n_2 + \ldots + n_l$ qudits\n
$H = H_{\text{in}} + H_{\text{proj}} + H_{\text{out}} + H_{\text{local}}$\n
$\phi$\n
1. $\{ \langle \psi | \langle \psi |_q = 0 \rangle \}$\n
$H_{\text{out}} = \{ \langle \psi | \langle \psi |_q = 0 \rangle \} \cup \{ \} \eta = 0$\n
$H_{\text{proj}} (\text{penalize illegal clock states})$\n
$H_{\text{in}}$ is $\mathcal{S}$-local\n
$H_{\text{proj}} (H_{\text{out}})$ are sums of projectors $\Pi_i$ ($\Pi_i = \Pi_i^2$)\n
If $\exists \phi$ such that $\langle \phi | H \phi \rangle = 1$ then $|\phi\rangle = \frac{1}{\sqrt{t}} \sum_{t=0}^{\infty} U_{t-1} U_t |\phi\rangle$ is the ground state of $H$ with energy $\lambda(H) \leq \frac{\epsilon}{1 + t}$\n
History state\n
If $\forall \phi$, $\text{Prob (} q = 1 \text{)} \leq \epsilon$ then $\lambda(H) \geq \frac{C(t - \epsilon)}{\epsilon^3}$ for some constant $C$\n
QMA versus QMA$_1$ - in QMA$_1$ verifying circuit output 1 with certainty, in QMA verifying quantum circuit output with high probability\n
Quantum $6$-SAT is QMA$_1$-complete
quantum k-SAT: \( \prod_{j=1}^{M} \Pi_j \cdot \Pi_i \) projection \( \phi = \Omega (u^\alpha) \alpha > 0 \)

Decide whether \( \exists \phi \) such that \( \forall i: \Pi_i |\phi\rangle = 0 \)

(i.e. \( H = \sum \Pi_i \) is frustration-free)

or \( \forall \phi \quad \langle \phi | \sum \Pi_i |\phi\rangle \geq 6 \)

"Can we find a state \( \phi \) which satisfies all constraints given by the projected \( \Pi_i \)?"

Quantum 5-SAT is \( \text{QMA}^+_\text{full} \)-complete. (later: quantum 2-SAT is easy)

Examples of frustration-free and non-frustration-free Hamiltonians:

\[ A_{\text{not FF}} \]

Ising ferromagnet is FF.

Quantum dimer model at Rokhsar-Kivelson point \( v = t \) is FF

\[ H = \sum_v \left( |\square\rangle \langle \square| + |\square| \langle \square| \right) - t \left( |\square\rangle \langle \square| + h.c. \right) \]

At \( v = t \) \( H = \sum_\square H_0 \quad \Pi_0 \) projector \( |\phi_0\rangle \propto \sum_\square |c\rangle \)

\[ H |\phi_0\rangle = 0 \]
Do these hardness results ("QMA-completeness") persist for Hamiltonians on 2D lattices?

What about classical spin systems?

Barahona '82. Planar graph $G = (V,E)$

Finding min. value of $H = \sum_{ij \in E} S_i S_j + \sum_{i \in V} S_i$

is a hard problem (is NP-complete).

Contrast: if graph $G$ is "1-dimensional", there is an efficient algorithm (in $|V| = n$) to find min of $H = \sum_{ij \in E} J_{ij} S_i S_j + \sum_{i \in V} J_i S_i$

$J_{ij}, J_i$ (random) fixed couplings.

"1D Icing spin glass."

Technique of dynamic programming

\[ \sigma = (1) \text{ in Ising spin.} \]

\[ H_{\text{opt}}(G_{ij} = S) = \min \left( H_{\text{opt}}^f(G_{ij} = S') + H_{r+1}(S', S) \right) \]

Optimal value of Hamiltonian on spins $r \ldots r+1$, when spin $G_{r+1}$ is set to $S$. 

\[ H_{r+1}(G_{ij} = S) = \sum_{i} \tilde{H}_{r+1}(G_{ij} = S') + H_{r+1}(S', S) \]
Recur in \( r \):

- Store \( H_2^{opt}(s_2 = s) \) for \( 2^m \) values of \( s \).
- Compute \( H_3^{opt}(s_3 = s) \) for \( 2^m \) values of \( s \) by executing minimum in \( S \) and using precomputed values \( H_2^{opt}(s_2 = s) \).
- Store \( H_3^{opt}(s_3 = s) \) etc.

How many steps (time)? \( n \cdot 2^m \cdot 2^m \cdot T(\text{eval } H_2, r, \langle r', s \rangle) \leq n \cdot 2^{m+1} \cdot m \)

- Efficient in \( n \).
- Not efficient when block spin is large.
- Dynamic programming can be done on a tree \( \checkmark \)
  (same idea: store optimum for partial trees given a boundary configuration)
- Or on "tree-like graphs" (graphs with bounded tree width)

Generalize to 1D quantum systems?

Chain of \( d \)-dimensional spins (qudits):

Product-state ansatz for ground-state \( |e_1 \rangle \otimes |e_2 \rangle \ldots |e_n \rangle \)

Choose a discrete set of \( K \) (\( k \geq d \)) states for each qudit.

Dynamic program finds optimum in time \( T \approx n K^2 T(\text{eval} H_3, \langle i \rangle) \)

- "Fat world"

How many (and which) states to choose?

Assume \( |f_0 \rangle = |e_0 \rangle \) but closest chosen state is \( |e_2 \rangle \langle e_2 | \otimes |e_2 \rangle \langle e_2 | \).

Then \( \lambda_{0 - \text{appro}} = 0 (\text{eq}) \).
In $d$-dimension how many steps $k$ to approximate any state $\rho$ with error $\epsilon$.

Approximately $k \sim (\frac{C}{\epsilon})^{2d}$ with constant $C$.

Set $\delta = C\epsilon$ for error $\delta$ in energy.

Then runtime $T \sim n \left(\frac{C\epsilon}{\delta}\right)^{2d}$ exponential in $d$.

if we rigorously want to control the error.

Schuch/Venturelli 2008: dynamic programming can be used

Abanov/Andriyash/Imani 2010 to optimize over HPS with constant bond dimension $D$.

rough idea: calculate mix energy of partial chain "fixing matrix" in HPS

$\left(\sum_i \text{Tr}(A_i \cdots A_n) (a_i \cdots a_n)\right)_{\text{at boundary}}$

Guaranteed to not get stuck in local minima, but

running time $T \sim n K^2 D^2 d^2$

bad dimension $A \times D \times D$ matrix

will be exponential in bond dimension $D$.

$\rightarrow$ so guaranteed bound on error, but impractical.

$\rightarrow$ no hope of obtaining $T \sim \text{poly}(D) \text{poly}(n) \text{poly}(\frac{1}{\epsilon})$

with $\epsilon$ error in energy estimate:

Schuch/Cirac/Venturelli 2008

class of $T \times T$ Hamiltonian with unique HPS ($D \leq \text{poly}(n)$)

where finding this HPS is at least as hard as factoring
Geometric locality for QMA-complete problems

Aharonov/Itaie/Kempe/Gottesman (AGIK)

1-DIM 12-state $H_1$ is QMA-complete

$$d=10 \quad H = \sum_{i=1}^{12} \sigma_i$$

"Had to determine eigenvalue with accuracy $\frac{1}{n}$".

AGIK result contrasts with

1. Eff. algorithm for any 10 classical system.

2. Suggests that not all 10-ground-states can be well represented by MPS with constant or even $\text{poly}(n)$-bond dimension $D$. (Optimize over MPS with $\text{poly}(n)$ bond dimension is in NP).

Physicist's criticism: "But asking for accuracy $\frac{1}{n}$ is very strong, $\lambda(n)$ scales extensively, i.e., as $\lambda(n) \sim n$. Why not focus on estimating $\lambda(n)$ with say, constant error?"

Complexity theorist: "But, there is always an eff. (in $n$) algorithm which does this, so I won't be able to prove that it is hard."
Homer Simpson's ground-state energy estimate

- Divide lattice of spins up in LxL blocks
- Omit terms in Hamiltonian which act between the blocks
- Do exact diagonalization (ED) in each LxL block, L = O(1) does not grow with system size.

$$||H - \sum \text{blocks}|| = O\left(\frac{n}{L^2}, L\right) = O\left(\frac{n}{L}\right).$$

Thus \( \frac{\lambda(H)}{n} - \frac{\sum \lambda(H\text{block})}{n} \leq \frac{1}{L} \equiv \epsilon \)

Procedure scales off in \( n \), but inefficient in \( L = \frac{1}{\epsilon} \) since we have to diagonalize \( L^2 \) matrix.

Prior to AG1K it was proved that

(Oliveira, Terhal 2005) 2-DIM 5-LOCAL (qubit) Hamiltonian is QMA-complete

and (using perturbation theory) that

2-DIM 2-LOCAL (qubit) Hamiltonian is QMA-complete (qubit)

extensions/simplifications abound, using perturbation theory.
Physics: \( H = H_0 + \lambda V \)  

Complexity: \( \text{Construct } H = H_0 + \lambda V \)  

Theory: \( \text{Heff describing } \)  

- Heff describing low-energy sector  
  (e.g. Heff is Hubbard model for \( U \gg t \))  
  Heff is therefore model)  

\[ \Rightarrow \text{Heff can be many-body when it is far-body.} \]  

E.g.: Heff is 5-local Hamiltonian whose ground state every problem is hard.  

\( H \) is 2-local Hamiltonian with same ground state.

Sketch how we modify Kitaev's quantum circuit-to-Hamiltonian construction to get a Hamiltonian on a 2D lattice.

Kitaev's Hamiltonian: \( H = H_{\text{int}} + H_{\text{prop}} + H_{\text{out}} + H_{\text{stab}} \) from circuit with \( U_1, \ldots, U_6 \), \(- U_\otimes H_\) \( t - 1 \) \( \ldots \)

\( k \)-local interactions on a 2D-lattice for some constant \( k \): each qubit is acted upon by \( O(1) \) number of terms \( t_i \). 

In quantum circuit, each qubit may be involved in many (scale with \( L \) gates from \( U_1, U_6 \), so it is not on a 2D lattice.

\[ \begin{align*} 
(10, 4) & \Rightarrow \text{Modify this quantum circuit so its functionality is the same, but qubits lie in a 2D plane.} 
\end{align*} \]
1. Change circuit to a circuit applied to qubits on a 1D line where only nearest-neighbor qubits can interact. (can be done by swapping qubits around, at most \( n \) swaps before every original gate).

   \[ \Rightarrow \text{ more gates } U_1 \rightarrow U_{\text{target}} \] Relabel as \( U_1 \rightarrow U_1 \).

2. Begin.

   Make \((2L-1)\) columns of \( n = n_1 + n_2 \) qubits in circuit at 0.

   Total of \((2L-1)\) \( (n_1 + n_2) \) qubits.

   First column, leave \( n_2 \) inputs free, set other \( n_1 \) inputs to \( 100 \cdots \).

   Other columns: all qubits are set to \( 100 \cdots \).

   Robot \( \swarrow \) (cursor) walks down odd column of qubits & walks up even column of qubits.

   In odd columns, robot implements \( U_i \) on qubits in column \( i \) when it passes those qubits.

   In even columns, robot swaps qubits in column \( i-1 \) with qubits in column \( i \) when it passes them.

   Robot moves active part of computation forward and executes gates.

At end, last column contains final state of computation.

Assume output qubit \( q \) is at the bottom.
Apply Kitaev's circuit-to-Hamiltonian construction to circuit at ②.

Robot clock state [1111000] where these clock qubits are laid on top of 2D array.

Thus term such as $|H \otimes |t\rangle \langle t-1|$ is 5-local geometrically-local term on lattice.

Here $\otimes$ is geometrically local since output qubit is near final $|0\rangle |0\rangle \otimes |t| |t+1\rangle$ state for cursor. ✓

Holes as before. ✓

$H_{in} = \sum_{|i\rangle} |i\rangle \langle i| \otimes |H=0\rangle \langle t=0|$

At $t=0$, cursor is far from qubit in most column, and from qubits in first column which are not close to the top.

Solution: So helpful qubits should be set to 10 only when cursor is about to pass them by.

One can work through Kitaev's proof and show that:

if $\chi(H) = 0$ it is 5-local on a 2D-lattice.

if $\chi(H) \geq \text{poly}(\epsilon)$
quantum 5-SAT is hard, (even quantum 4-SAT on a 20 lattice) is hard.

What is the relation with classical SAT \((x_1 \lor x_2 \lor x_3) \land (x_4 \lor x_5 \lor \ldots)\) clause.

Each clause disallows one configuration.

\[ x_1 \lor x_2 \lor x_3 = 1 \]

\[ x_1 = 0, x_2 = 1, x_3 = 0 \] is not allowed.

\[ \Pi = |0\rangle\langle 0|_{x_1} \otimes |1\rangle\langle 1|_{x_2} \otimes |0\rangle\langle 0|_{x_3} \]

if \( \Pi \psi = \psi \)

2-dim space \( \Pi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) rank 1.

Classical SAT • projectors are diagonal in standard basis (spin) • projectors are of rank 1.

\( q(x) = 1 \) iff \( \exists \psi \) such that \( \Pi_i \psi = |\psi\rangle \) for all \( \Pi_i \)

\( |\psi\rangle \) is of course some bitstring \( |x_1 \cdots x_3 \cdots x_i \cdots \rangle \)

Classical 3-SAT is NP-complete (and 2-SAT was easy).

What about quantum 2-SAT?

Quantum 1-SAT \( \in P \) (Similar to classical 1-SAT)

Bravyi: quantum 2-SAT \( \not\in P \)

2006.

Chen et al. (2010): if quantum 2-SAT has a SATifying assignment \( \Psi \), it is a product of 2 qubit and 1 qubit states.

easy \[ \xrightarrow{\text{hard for}} \] quantum computer

for classical computer
Sketch of Bravyi's algorithm (rigorous form of real-space renormalization)

Given is a set of $n$ qubits, each $T_i$ acts on 2 qubits.
(no geometric constraint)

Alternately, $H = \sum_i T_i$ and $T_i$ is projector onto lowest eigenvectors of $H_i$.

Decide whether $\exists \psi \text{ s.t. } \forall i : T_i |\psi\rangle = 0$

($\forall i \exists \psi \text{ s.t. } \forall i : T_i |\psi\rangle = 0$; $H$ is frustration-free)

Approximation method for finding state with low eigenvalues:

Take $T_i$ as projector onto low-lying (but not necessarily lowest) eigenstates of $H_i$.

Case 1: All $T_i$ have rank 1 ⇒ problem is in special form (like torus
and proceed by combining constraints...)

Case 2: for $i = 1, \ldots, n$, check rank ($T_i$):

- rank ($T_i$) = $n$ or $T_i = I$ then all $T_i$ are Rank-1 operators.
- rank ($T_i$) = 3, or $T_i = I - |ab\rangle \langle ba|$, only state $|ab\rangle$ is allowed. (keep states a and b)

Set $|\psi_{nk}\rangle = |ab\rangle \otimes |\psi'\rangle$

Repeat other $T_i$ which act on qubits a or b.

One can show that:

- Any $T_i$ that is replaced by a 1-qubit constraint on qubit $d$
- Any $T_i$ is left the same, of course.
\[ \text{rank}(\Pi_i) = 2, \text{ 2-dim space is allowed.} \]

\[ V^+V = I \]

but \( \Pi^+ = \text{Projector onto allowed space} \)
\( V \) is a isometry (not unitary)

\[ \Pi_{ab} = I - \Pi^+ \]

Hence 9 qubits 4 of into one qutrit, say c \((n-1) \text{ qubits})\)
and rewrite other constraints

\[ \Pi_{ab} \text{ 11.5 stays the same} \]

\[ Q_{cij} = V^+ \Pi_{a},g, V_\ell \]

Set  \( \text{if } \exists_{ij}, Q_{cij} |14 \rangle = 0 \)

\[ \rightarrow Q_{cij} \text{ need not be a projector (but take projector onto null-space and proceed).} \]

\[ \text{rank}(Q_{cij}) \leq \text{rank}(\Pi_{a},g) \]
null space may be bigger than before.

So we eliminate rank \( \Pi_i = 2 \) & rank \( \Pi_i = 3 \) constraints
and reduce rank of other constraints.

Entanglement Renormalization\( \text{ Keep going until we only have rank 1 constraints }\)
and fewer qubits.

\[ \text{Let not keep going } \text{ rank}(\Pi_i) = 2 \text{, replace 2 qubits by a qutrit (d=3)} \]
9-dim space so if rank \( \Pi \) stays the same,
next round needs 9-rank \( \Pi \) dim. space.

bigger and bigger qutrits.
We are back in Case 1; all \( T_i \) have rank 1.

Combine rank 1 constraints

**Lemma** Assume \( T_b \langle f \rangle = 0 \), \( T_a = |4ab\rangle \langle 4ab| \)

and \( T_c \langle f \rangle = 0 \), \( T_e = |6bc\rangle \langle 6bc| \).

Then also \( T_{ac} \langle f \rangle = 0 \) with \( T_{ac} = |4bc\rangle \langle 4bc| \).

where \( |4bc\rangle \propto \langle 4|-4ab, 6bc\rangle \text{ singlet} \)

(Verify yourself)

All constraints can be combined with old constraints \( T_{ac} \).

i.e. sometimes we now only have a 2, 1 or 0-dim allowed space for qubits a and c.

So either we have fewer qubits, or we have checked all triples \( a, b, c \) \((O(n^3) \text{ triples})\) for mutual consistency.

Complete set of constraints:

Bravyi shows that product state is a set for such system.

2-SAT on qutrits? How to generalize the Lemma in which we combine constraints.

Perhaps rigorous results can be obtained for restricted models (\( SU(2) \)-symmetric e.g.)